NONLINEAR EVOLUTION EQUATIONS IN BANACH SPACES

BY

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ABSTRACT

The evolution problem $0 \in du/dt + A(t)u(t)$, $u(s) = x$, where the $A(t)$ are nonlinear operators acting in a Banach space, is studied. Evolution operators are constructed from the $A(t)$ under various assumptions. Basic properties of these evolution operators are established and their relationship to the evolution equation is determined. The results obtained extend several known existence theorems and provide generalized solutions of the evolution equation in more general cases.

Introduction

Let X be a Banach space and consider the initial value problem

$$(1)\qquad \begin{cases} \dfrac{du}{dt} + A(t)u \ni 0, & s \leqq t \leqq T \\ u(s) = x \end{cases}$$

for an X-valued function u, where, for each t, $A(t)$ is a nonlinear (and possibly multivalued) operator. Suppose for the moment that the problem (1) has a unique solution on $[s, T]$ for every $x \in X$ and s in $[0, T]$. Defining the operator $U(t,s)$ by $U(t,s)x = u(t)$, where $u(t)$ is the solution of (1), we immediately obtain the relations

(i) $U(s,s) = I$ (the identity operator), and $U(t,s)U(s,r) = U(t,r)$ for $0 \leqq r \leqq s \leqq t \leqq T$

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from the assumed uniqueness of solutions of (1). One also will have that $U(t,s)x$ is continuous in t for fixed s and x under most definitions of a "solution" of (1). Usually there is a stronger continuity present, namely:

(ii) $U(t,s)x$ is continuous in the pair (t,s) on the triangle $0 \leqq s \leqq t \leqq T$.

A family of operators $U(t,s)$ satisfying (i) and (ii) is called an *evolution operator.* If the domain of each $U(t,s)$ is a subset C of X (rather than X), then we will say U is an evolution operator on C. If U arises from (1) as sketched above, we will call it the evolution operator for the problem (1) or the evolution operator associated with $A(t)$. The main goal of this work is to study various sets of conditions on $A(t)$ which are sufficient to guarantee that there is an evolution operator associated (perhaps in a generalized sense) with $A(t)$. Our assumptions will restrict us to evolution operators U such that there is a number ω satisfying

$$\| U(t,s)x - U(t,s)y \| \leqq e^{\omega(t-s)}\| x - y \| \tag{2}$$

for $0 \leqq s \leqq t \leqq T$ and x, y in the domain of U.

If X is a Hilbert space, then one can actually characterize those operators A which are independent of t and give rise to evolution operators U satisfying (2). (In this case $U(t,s)$ depends only on $t-s$ and so defines a semigroup.) See [20] and [11]. Moreover, if X is a general Banach space, then the Hille-Yosida Theorem (see, e.g., [13], [22]) provides a complete characterization in the linear t-independent case. However, no such characterization is known corresponding to the case of linear t-dependent $A(t)$. There are numerous sets of sufficient conditions assuring the existence of an evolution operator associated with $A(t)$ in the linear case. A discussion of results of this type and references may be found in [22, ch. XIV].

Some results are available for the quasi-autonomous case (i.e., equations of the form (3) below) under the assumption that X^* is uniformly convex. See [15] and [3]. The paper [14] allows a more general t-dependence but requires uniform convexity of X^* and $A(t)$ is assumed single-valued. See [6], [7] and [17] also.

Very little is known when X is not restricted and $A(t)$ is nonlinear and t-dependent. The case of continuous $A(t)$ is treated in [16]. See [21] also. In this work we will give conditions on $A(t)$ guaranteeing the existence of an associated evolution operator. These conditions will include as special cases many of those used in the above quoted works. The arguments and results generalize those of the paper [10] in which the autonomous case was treated. The operator $U(t,s)$ is constructed in Section 2 from the $A(t)$ via a product formula and continuity properties of $U(t,s)$ are studied. Section 3 is devoted to the solution of the initial value problem (1). It is shown that $U(t,s)x$ is a solution of (1) in the usual sense if (1) has such a solution. Otherwise, $U(t,s)x$ should be regarded as a generalized solution. Section 4 deals with the convergence of a family $U^{\beta}(t,s)$ of evolution operators when the corresponding $A^{\beta}(t)$ are known to converge, in some sense, as $\beta \to 0$. These results are then used to approximate a given evolution operator $U(t,s)$ by means of differentiable evolution operators obtained by solving problems which approximate (1). The results of this section generalize those of [5] for the autonomous case. Section 5 deals with the quasi-autonomous case, i.e., the problem

$$\text{(3)} \qquad \begin{cases} \dfrac{du}{dt} + Au \ni f(t) \\ u(s) = x \end{cases}$$

where A is t-independent. The existence theorems of Kato [14] are extended to any reflexive Banach space. Generalized solutions of (3) are obtained for $f \in L^1([0,T]: X)$ and arbitrary X by a straightforward extension of the ideas of [1]. Section 6 is concerned with the problem

$$\begin{cases} \dfrac{du}{dt} + A(t)u \ni 0 \\ u(0) = u(T) \end{cases}$$

The results obtained extend some work of Brezis in [3] concerning the quasi-autonomous case in Hilbert space. Section 7 is devoted to a simple application of our basic existence theorems to a concrete evolution problem in partial differential equations.

1. Preliminaries on accretive sets

In this section we collect some basic definitions and elementary facts. Many of the results are standard and appear in the existing literature. See, e.g. [10], [14], [15] and [19].

Let X be a real Banach space with the norm $\| \;\|$. A subset A of $X \times X$ is in the class $\mathscr{A}(\omega)$ if for each $\lambda > 0$ such that $\lambda\omega < 1$ and each pair $[x_i, y_i] \in A$, $i = 1,2$, we have

$$\|(x_1 + \lambda y_1) - (x_2 + \lambda y_2)\| \geqq (1 - \lambda\omega)\; \|x_1 - x_2\|$$

A is called *accretive* if $A \in \mathscr{A}(0)$. If λ is real, J_λ will denote the set $(I + \lambda A)^{-1}$ and

$D_\lambda = D(J_\lambda) = R(I + \lambda A)$. ($D(J_\lambda)$ is the domain of J_λ and $R(I + \lambda A)$ is the range of $I + \lambda A$).

LEMMA 1.1. *Let ω be real and $A \in \mathscr{A}(\omega)$. If $\lambda \geqq 0$ and $\lambda\omega < 1$, then the following statements hold:*

(i) *J_λ is a function and for $x, y \in D_\lambda$*

$$\| J_\lambda x - J_\lambda y \| \leqq (1 - \lambda\omega)^{-1} \| x - y \|.$$

(ii) *If $x \in D_\lambda \cap D(A)$ we have*

$$\| J_\lambda x - x \| \leqq \lambda(1 - \lambda\omega)^{-1} \inf_{y \in Ax} \| y \|$$

(iii) *If $\omega \geqq 0$, $n \geqq 1$ is an integer and $x \in D(J_\lambda^n)$, then*

$$\| J_\lambda^n x - x \| \leqq n(1 - \lambda\omega)^{-n+1} \| J_\lambda x - x \|$$

(iv) *If $x \in D_\lambda$ and $\lambda \geqq \mu > 0$, then*

$$\frac{\mu}{\lambda} x + \frac{\lambda - \mu}{\lambda} J_\lambda x \in D_\mu$$

and

$$J_\lambda x = J_\mu \left(\frac{\mu}{\lambda} x + \frac{\lambda - \mu}{\lambda} J_\lambda x \right) \tag{1.1}$$

Equation (1.1) is called the *resolvent identity* or *equation*. A proof of Lemma 1.1 is given in [10, lemma 1.2]. For λ, A satisfying the conditions of Lemma 1.1, A_λ will denote the function $\lambda^{-1}(I - J_\lambda)$. Concerning A_λ we have:

LEMMA 1.2. *Let $A \in \mathscr{A}(\omega)$, $\lambda > 0$, $\lambda\omega < 1$. Then the following statements hold:*

(i) *If $x \in D_\lambda \cap D_\mu$ and $0 < \mu \leqq \lambda$, then*

$$(1 - \lambda\omega) \| A_\lambda x \| \leqq (1 - \mu\omega) \| A_\mu x \|$$

(ii) *If $x, y \in D_\lambda$ then*

$$\| A_\lambda x - A_\lambda y \| \leqq \lambda^{-1}(1 + (1 - \lambda\omega)^{-1}) \| x - y \|$$

(iii) $$A_\lambda \in \mathscr{A}(\omega(1 - \lambda\omega)^{-1})$$

(iv) *If $x \in D_\lambda \cap D(A)$, then $\| A_\lambda x \| \leqq (1 - \lambda\omega)^{-1} \inf_{y \in Ax} \| y \|$.*

PROOF. The assertions (ii) and (iv) follow at once from Lemma 1.1 (i) and (ii) respectively. To prove (i), we use Lemma 1.1 (iv) and (i) to conclude

$$\begin{aligned}\| A_\lambda x \| &= \lambda^{-1}\| x - J_\lambda x \| \leqq \lambda^{-1}(\| x - J_\mu x \| + \| J_\mu x - J_\lambda x \|) \\ &\leqq \frac{\mu}{\lambda}\| A_\mu x \| + \frac{1}{\lambda} \left\| J_\mu x - J_\mu \left(\frac{\mu}{\lambda} x + \frac{\lambda - \mu}{\lambda} J_\lambda x \right)\right\| \\ &\leqq \frac{\mu}{\lambda}\| A_\mu x \| + \frac{\lambda - \mu}{\lambda}(1 - \mu\omega)^{-1} \left\| \frac{x - J_\lambda x}{\lambda} \right\| \\ &= \frac{\mu}{\lambda}\| A_\mu x \| + \frac{\lambda - \mu}{\lambda}(1 - \mu\omega)^{-1} \| A_\lambda x \|.\end{aligned}$$

Rearranging this inequality yields (i). To obtain (iii), let $\rho \geqq 0$. Lemma 1.1 (i) implies

$$\begin{aligned}\| x - y + \rho(A_\lambda x - A_\lambda y) \| &= \left\| \left(1 + \frac{\rho}{\lambda}\right)(x - y) - \frac{\rho}{\lambda}(J_\lambda x - J_\lambda y) \right\| \\ &\geqq \left(1 + \frac{\rho}{\lambda}\right) \| x - y \| - \frac{\rho}{\lambda} \| J_\lambda x - J_\lambda y \| \\ &\geqq \left(\left(1 + \frac{\rho}{\lambda}\right) - \left(\frac{\rho}{\lambda}\right)(1 - \lambda\omega)^{-1}\right) \| x - y \| \\ &= (1 - \rho\omega(1 - \lambda\omega)^{-1}) \| x - y \|,\end{aligned}$$

and the proof is complete.

The notions in the next definition are introduced and studied in [9].

DEFINITION 1.1. Let $A \in \mathscr{A}(\omega)$ and set $\mathscr{D} = \bigcup_{\kappa > 0} (\bigcap_{0 < \lambda < \kappa} D_\lambda)$. For $x \in \mathscr{D}$ set

$$|Ax| = \lim_{\lambda \downarrow 0} \| A_\lambda x \|. \tag{1.2}$$

If $\mathscr{D} \supseteq D(A)$, then

$$\hat{D}(A) = \{x \in \mathscr{D} : |Ax| < \infty\}. \tag{1.3}$$

We allow the possibility that $|Ax| = \infty$. Lemma 1.2 (i) then guarantees that the limit (1.2) exists if $x \in \mathscr{D}$. The set $\hat{D}(A)$ is an extension of the domain of A, since Lemma 1.2 (iv) shows $|Ax| < \infty$ if $x \in D(A) \cap \mathscr{D}$. If $D_\lambda = X$ for small positive λ and X is reflexive, then $D(A) = \hat{D}(A)$ and $|Ax| = \inf_{y \in Ax} \| y \|$ for $x \in D(A)$. This is not the case in general. See [9] and [18], In view of Lemma 1.2, we have:

LEMMA 1.3. *Let $A \in \mathscr{A}(\omega), \mathscr{D} \supseteq D(A)$, $\lambda > 0$ and $\lambda\omega < 1$. Then*

(i) $$\| A_\lambda x \| \leqq (1 - \lambda\omega)^{-1} |Ax| \text{ for } x \in D_\lambda \cap \mathscr{D}$$

and

(ii) $$|Ax| \leqq \inf_{y \in Ax} \|y\| \text{ for } x \in D(A).$$

The only other fact we need concerning $|Ax|$ is:

LEMMA 1.4. *Let $x_n \in D_\lambda$ for $n = 1, 2, \cdots$ and $0 < \lambda < \lambda^0$. Let $x_n \to x_0 \in D_\lambda$* as *$n \to \infty$. Then $|Ax_0| \leqq \liminf_{n\to\infty} |Ax_n|$. In particular, if $|Ax_n|$ is bounded, then $x_0 \in \hat{D}(A)$.*

PROOF: $$\|A_\lambda x_0\| \leqq \|A_\lambda x_n\| + \|A_\lambda x_n - A_\lambda x_0\|$$
$$\leqq (1-\lambda\omega)^{-1}|Ax_n| + \lambda^{-1}(1 + (1-\lambda\omega)^{-1}\|x_n - x_0\|$$

where we used Lemma 1.2 (ii) and Lemma 1.3 (i). The result follows upon letting $n \to \infty$ and then $\lambda \downarrow 0$.

2. Existence and properties of the evolution operator

In this section we construct the evolution operator $U(t,s)$ associated with a one parameter family $A(t)$ of $\mathscr{A}(\omega)$ operators (sets) and establish its main properties. Throughout this section T, ω denote real numbers, $T > 0$, and $A(t)$ satisfies

(A.1) $$A(t) \in \mathscr{A}(\omega) \quad \text{for } 0 \leqq t \leqq T.$$

(A.2) $\overline{D(A(t))} = \bar{D}$ is independent of t. (We choose, e.g., $D = D(A(0))$).

(A.3) $R(I + \lambda A(t)) \supset \bar{D}$ for $0 \leqq t \leqq T$ and $0 < \lambda < \lambda_0$, where $\lambda_0 > 0$ and $\lambda_0 \omega < 1$.

Let $J_\lambda(t) = (I + \lambda A(t))^{-1}$. The t-dependence of $A(t)$ will be restricted by one of the following two conditions:

(C.1) There is a continuous function $f: [0,T] \to X$ and a monotone increasing function $L: [0,\infty) \to [0,\infty)$ such that

(2.1) $$\|J_\lambda(t)x - J_\lambda(\tau)x\| \leqq \lambda \|f(t) - f(\tau)\| L(\|x\|) \text{ for } 0 < \lambda < \lambda_0, \quad 0 \leqq t, \tau \leqq T \text{ and } x \in \bar{D}.$$

(C.2) There is a continuous function $f: [0,T] \to X$ which is of bounded variation on $[0,T]$, and a monotone increasing function $L: [0,\infty) \to [0,\infty)$ such that

(2.2) $$\|J_\lambda(t)x - J_\lambda(\tau)x\| \leqq \lambda \|f(t) - f(\tau)\| L(\|x\|)(1 + |A(\tau)x|)$$

for $0 < \lambda < \lambda_0$, $0 \leqq t, \tau \leqq T$ and $x \in \bar{D}$. (f is of bounded variation on $[0,T]$ if there is a constant V such that $\sum_{i=1}^{n-1} \|f(t_{i+1}) - f(t_i)\| \leqq V$ whenever $0 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_n \leqq T$).

It follows from the conditions (A.1) to (A.3) that for each fixed τ, $0 \leqq \tau \leqq T$, there is a semigroup $S_\tau(t)$ on $\bar{D}$ associated with $A(\tau)$. See [10]. The condition (C.1) assumes only continuity of f and corresponds roughly to the case in which $A(t)$ has the form $A(0) + B(t)$, where $B(t)x$ is well-behaved in x. In particular, taking $B(t)x$ to be independent of x, we can treat the quasi-autonomous case in adequate generality. See Section 5 below. Condition (C.2) adds the requirement that f be of bounded variation, but weakens (2.1) to (2.2). Choosing $f(t) = tx_0$ for some $x_0 \in X$, $x_0 \neq 0$, (C.2) becomes a Lipschitz continuity condition which is implied by some assumptions used in the study of linear evolution equations. This special case was also treated by Kato in [14] under the additional assumptions that X^* is uniformly convex and $A(t)$ is single-valued.

An immediate consequence of either (C.1) or (C.2) is that $\hat{D}(A(t))$ is independent of t. More precisely, dividing (2.1) by λ and letting $\lambda \downarrow 0$ we see that

$$|A(t)x| \leqq |A(\tau)x| + \|f(t) - f(\tau)\| L(\|x\|), \tag{2.3}$$

while treating (2.2) similarly yields

$$|A(t)x| \leqq |A(\tau)x| + \|f(t) - f(\tau)\| L(\|x\|)(1 + |A(\tau)x|) \tag{2.4}$$

for $0 \leqq t$, $\tau \leqq T$ and $x \in \bar{D}$. We set $\hat{D} = \hat{D}(A(t))$ for $0 \leqq t \leqq T$.

The main result of this section is the following theorem.

THEOREM 2.1. *Let $A(t)$ satisfy* (A.1), (A.2) *and* (A.3). *If either* (C.1) *or* (C.2) *hold, then*

$$U(t,s)x = \lim_{n \to \infty} \prod_{i=1}^{n} J_{(t-s)/n}\left(s + i\left(\frac{t-s}{n}\right)\right)x \tag{2.5}$$

exists for $x \in \bar{D}$ and $0 \leqq s < t \leqq T$. The $U(t,s)$ defined by (2.5) *for $0 \leqq s < t \leqq T$ and by the identity for $0 \leqq s = t \leqq T$ is an evolution operator on $\bar{D}$. Moreover,*

$$\|U(t,s)x - U(t,s)y\| \leqq e^{\omega(t-s)}\|x - y\| \tag{2.6}$$

for $0 \leqq s,t \leqq T$ and $x, y \in \bar{D}$.

We use the conventions $\prod_{i=j}^{j} T_i = T_j$, $\prod_{i=j}^{k+1} T_i = T_{k+1}(\prod_{i=j}^{k} T_i)$ if $k \geqq j$ and $\prod_{i=j}^{k} T_i =$identy if $k<j$, where $\{T_i\}$ is any collection of functions. The proof of Theorem 2.1 is based on a technique of [10, appendix to sec. 1]. More precise information concerning the continuity properties of $U(t,s)$ is collected in the

course of proving Theorem 2.1 and will be stated in Propositions 2.1, 2.2 and 2.3. Three lemmas precede the main arguments.

LEMMA 2.1. *Let $n \geqq m > 0$ be integers and α, β be positive numbers satisfying $\alpha + \beta = 1$. Then*

$$\sum_{j=0}^{m} \binom{n}{j} \alpha^j \beta^{n-j} (m-j) \leqq [(n\alpha - m)^2 + n\alpha\beta]^{\frac{1}{2}} \tag{2.7}$$

and

$$\sum_{j=m}^{n} \binom{j-1}{m-1} \alpha^m \beta^{j-m} (n-j) \leqq \left[\frac{m\beta}{\alpha^2} + \left(\frac{m\beta}{\alpha} + m - n \right)^2 \right]^{\frac{1}{2}}. \tag{2.8}$$

See [10, lemma 1.4] for a proof of Lemma 2.1.

DEFINITION 2.1. For $x \in \bar{D}$ let

$$M(x) = \sup_{0 \leqq t \leqq T} |A(t)x|.$$

It follows directly from (2.3) and (2.4) that if either (C.1) or (C.2) hold, then $M(x) < \infty$ for every $x \in \hat{D}$.

LEMMA 2.2. *There exists a constant K depending only on T, λ_0 and ω such that if $x \in \hat{D}$, $l \geqq 0$, $0 \leqq s_i \leqq T$ for $i = 1, 2, \cdots, l$, $0 < \lambda < \lambda_0$, and $\lambda l \leqq T$, then*

$$\left\| \prod_{i=1}^{l} J_\lambda(s_i)x - x \right\| \leqq K l \lambda M(x). \tag{2.9}$$

PROOF.

$$\left\| \prod_{i=1}^{l} J_\lambda(s_i)x - x \right\| = \left\| \sum_{k=1}^{l} \left(\prod_{i=k}^{l} J_\lambda(s_i)x - \prod_{i=k+1}^{l} J_\lambda(s_i)x \right) \right\|$$

$$\leqq \sum_{k=1}^{l} (1 - \lambda\omega)^{-l+k-1} \lambda |A(s_k)x| \leqq N l \lambda M(x)$$

where $N = \max\{(1 - \lambda\omega)^{-j} : 0 \leqq j \leqq l\}$ and we used Lemma 1.1 (i), Lemma 1.3 (ii) and the definition of A_λ. If $\omega \leqq 0$, we take $K = 1 \geqq N$. If $\omega \geqq 0$, we use the elementary estimate

$$(1 - \gamma)^{-n} \leqq \exp\left(\frac{n\gamma}{1 - \gamma} \right) \text{ for } n \geqq 0, \ 0 \leqq \gamma < 1$$

to obtain

$$N = (1 - \lambda\omega)^{-l} \leqq \exp\left(\frac{l\lambda\omega}{1 - \lambda\omega}\right) \leqq \exp\left(\frac{T\omega}{1 - \lambda_0\omega}\right), \tag{2.10}$$

and set $K = \exp(T\omega/(1 - \lambda_0\omega))$.

Throughout the rest of this paper we will use the following notation

$$P_{\lambda,k}(s)x = \prod_{i=1}^{k} J_\lambda(s + i\lambda)x. \tag{2.11}$$

When there is no danger of confusion we will write $P_{\lambda,k}$ instead of $P_{\lambda,k}(s)x$.

LEMMA 2.3. *Let* (C.2) *hold and* $x \in \hat{D}$. *Then there is a constant C depending only on* $\| x \|$, $M(x)$, λ_0, ω *and T such that*

$$M(P_{\lambda,l}(s)x) \leqq C \tag{2.12}$$

whenever $0 < \lambda \leqq \lambda_0$, $l \geqq 0$, $0 \leqq s \leqq T - l\lambda$.

PROOF. Set $a_j = | A(s + j\lambda)P_{\lambda,j}(s)x |$. Since $A_\lambda(s + j\lambda)P_{\lambda,j-1} \in A(s + j\lambda)P_{\lambda,j}$ we have, by Lemma 1.3 (i), (ii) and (2.4)

$$\begin{aligned} a_j &= | A(s + j\lambda)P_{\lambda,j} | \leqq \| A_\lambda(s + j\lambda)P_{\lambda,j-1} \| \leqq (1 - \lambda\omega)^{-1} | A(s + j\lambda)P_{\lambda,j-1} | \\ &\leqq (1 - \lambda\omega)^{-1}\{ | A(s + (j - 1)\lambda)P_{\lambda,j-1} | \\ &\quad + \| f(s + j\lambda) - f(s + (j - 1)\lambda) \| L(\| P_{\lambda,j-1} \|)(1 + | A(s + (j - 1)\lambda)P_{\lambda,j-1} |)\} \\ &\leqq (1 - \lambda\omega)^{-1}(a_{j-1} + b_j(1 + a_{j-1})) \end{aligned}$$

where

$$b_j = \| f(s + j\lambda) - f(s + (j - 1)\lambda) \| L(\| x \| + KTM(x)) \tag{2.13}$$

and we used Lemma 2.2 to estimate $\| P_{\lambda,j-1} \|$. Setting $d_j = (1 - \lambda\omega)^{-1}b_j$ and $c_j = (1 - \lambda\omega)^{-1}(1 + b_j)$, we have the recursive estimate

$$a_j \leqq c_j a_{j-1} + d_j. \tag{2.14}$$

Now (2.14) implies

$$a_l \leqq \left(\prod_{i=1}^{l} c_i\right)a_0 + \sum_{j=1}^{l} \left(\prod_{i=j+1}^{l} c_i\right)d_j.$$

It suffices to treat the case $\omega \geqq 0$. Recalling the definition of d_j and c_j and using

$$\prod_{i=j}^{l} (1 + b_j) \leqq \exp\left(\sum_{i=j}^{l} b_i\right) \leqq \exp\left(\sum_{i=1}^{l} b_i\right),$$

we obtain

$$a_l \leqq (1-\lambda\omega)^{-l}\exp\left(\sum_{i=1}^{l} b_i\right)\left\{a_0 + \sum_{i=1}^{l} b_i\right\}. \tag{2.15}$$

Next, $a_0 \leqq M(x)$ and if V is the variation of f over $[0,T]$, then (2.13) gives

$$\sum_{i=1}^{l} b_i \leqq VL(\|x\| + KTM(x)).$$

Thus (2.15) provides a bound on a_l. Finally, $|A(t)P_{\lambda,l}(s)x|$ is easily estimated in terms of $a_l = |A(s+l\lambda)P_{\lambda,l}(s)x|$ via (2.4). The proof is complete.

Let $f(t)$ be the function introduced in the conditions (C.1) and (C.2) and

$$\rho(r) = \sup\{\|f(t)-f(\tau)\| : 0 \leqq t, \tau \leqq T \text{ and } |t-\tau| \leqq r\}, \tag{2.16}$$

i.e., ρ is the modulus of continuity of f on $[0,T]$. Clearly, $\rho\colon [0,\infty) \to [0,\rho(T)]$ is nondecreasing, $\lim_{r\to 0}\rho(r) = \rho(0) = 0$, and ρ is subadditive, i.e., $\rho(r+s) \leqq \rho(r) + \rho(s)$ for $r,s \geqq 0$.

Proof of Theorem 2.1. Let $x \in \hat{D}$ and $\lambda_0 \geqq \lambda \geqq \mu > 0$. Set $a_{k,l} = \|P_{\lambda,k}(s)x - P_{\mu,l}(s)x\|$ where $k,l \geqq 0$. We have, for $k,l \geqq 1$, $a_{k,l} = \|J_\lambda(s+k\lambda)P_{\lambda,k-1} - J_\mu(s+l\mu)P_{\mu,l-1}\| \leqq \|J_\lambda(s+k\lambda)P_{\lambda,k-1} - J_\mu(s+k\lambda)P_{\mu,l-1}\| + \|J_\mu(s+k\lambda)P_{\mu,l-1} - J\mu(s+l\mu)P_{\mu\, l-1}\|$. Using Lemma 1.1 and the resolvent identity we obtain

$$\begin{aligned}
&\|J_\lambda(s+k\lambda)P_{\lambda,k-1} - J_\mu(s+k\lambda)P_{\mu\, l-1}\| \\
&= \left\|J_\mu(s+k\lambda)\left(\frac{\mu}{\lambda}P_{\lambda,k-1} + \frac{\lambda-\mu}{\lambda}J_\lambda(s+k\lambda)P_{\lambda,k-1}\right) - J_\mu(s+k\lambda)P_{\mu,l-1}\right\| \\
&\leqq (1-\mu\omega)^{-1}\left[\frac{\mu}{\lambda}a_{k-1,l-1} + \frac{\lambda-\mu}{\lambda}a_{k,l-1}\right].
\end{aligned}$$

Hence for $k,l \geqq 1$ we have

$$a_{k,l} \leqq \alpha_1, a_{k-1,l-1} + \beta_1 a_{k,l-1} + b_{k,l} \tag{2.17}$$

where

$$\alpha_1 = \alpha(1-\mu\omega)^{-1},\ \beta_1 = \beta(1-\mu\omega)^{-1},\ \alpha = \mu/\lambda, \quad \beta = (\lambda-\mu)/\lambda$$

and

$$b_{k,l} = \|J_\mu(s+l\mu)P_{\mu,l-1} - J_\mu(s+k\lambda)P_{\mu\, l-1}\|. \tag{2.18}$$

It is proved in Appendix 1 that (2.17) implies

$$
\begin{aligned}
a_{m,n} \leqq & \sum_{i=0}^{(m-1)\wedge n} \beta_1^{n-i}\alpha_1^i \binom{n}{i} a_{m-i,0} + \sum_{i=m}^{n} \alpha_1^m \beta_1^{i-m} \binom{i-1}{m-1} a_{0,n-i} \\
& + \sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \beta_1^{j-1}\alpha_1^i \binom{j}{i} b_{m-i,n-j}
\end{aligned} \tag{2.19}
$$

where $l \wedge k = \min(l, k)$, $\binom{j}{i}$ is the binomial coefficient and $m, n \geqq 0$. It follows from Lemma 2.2 that

$$
\begin{cases} a_{l,0} \leqq Kl\lambda M(x) \\ a_{0,l} \leqq Kl\mu M(x). \end{cases} \tag{2.20}
$$

Using (2.19), (2.20) and Lemma 2.1 we deduce

$$
\begin{aligned}
a_{m,n} \leqq & \; KM(x)\{[(n\mu - m\lambda)^2 + n\mu(\lambda - \mu)]^{\frac{1}{2}} \\
& + [(n\mu - m\lambda)^2 + m\lambda(\lambda - \mu)]^{\frac{1}{2}}\} + \sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \beta_1^{j-1}\alpha_1^i \binom{j}{i} b_{m-i,n-j},
\end{aligned} \tag{2.21}
$$

where K depends only on T, ω and λ_0, while m, n are restricted by $m\lambda, n\mu \leqq T - s$. We next obtain estimates on the $b_{k,l}$ under assumptions (C.1) and (C.2). If (C.2) holds, then (2.2) implies

$$
\begin{aligned}
b_{k,l} &= \| J_\mu(s + l\mu)P_{\mu,l-1} - J_\mu(s + k\lambda)P_{\mu,l-1} \| \\
&\leqq \mu \| f(s + l\mu) - f(s + k\lambda) \| L(\| P_{\mu,l-1} \|)(1 + | A(s + k\lambda)P_{\mu,l-1} |).
\end{aligned}
$$

Next, Lemma 2.2 provides a bound on $\| P_{\mu,l-1} \|$ and Lemma 2.3 provides one for $| A(s + k\lambda)P_{\mu,l-1} |$. Therefore there is a K (we use K to denote various constants) such that

$$
b_{k,l} \leqq K\mu \| f(s + l\mu) - f(s + k\lambda) \|. \tag{2.22}
$$

An estimate of the form (2.22) will also hold under (C.1) instead of (C.2). The argument is similar to the above, but requires only Lemma 2.2. It follows that under either (C.1) or (C.2) we have a constant K such that

$$
\begin{aligned}
&\sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \beta_1^{j-i}\alpha_1^i \binom{j}{i} b_{m-i,n-j} \\
&\leqq K\mu(1-\mu\omega)^{-n} \left(\sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \beta^{j-i}\alpha^i \binom{j}{i} \rho(|(n-j)\mu-(m-i)\lambda|) \right) \\
&\leqq K\mu(1-\mu\omega)^{-n} \Big[n\rho(|n\mu-m\lambda|) \\
&\quad + \sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \beta^{j-i}\alpha^i \binom{j}{i} \rho(|j\mu-i\lambda|) \Big] .
\end{aligned} \tag{2.23}
$$

The first inequality in (2.23) employed (2.22) and the definitions of α_1, β_1 and ρ. The second inequality used the subadditivity of ρ and the estimate

$$\sum_{i=0}^{(m-1)\wedge j} \beta^{j-i}\alpha^i \binom{j}{i} \leqq 1.$$

Next let $\delta > 0$ be given. Write

$$\sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \beta^{j-i}\alpha^i \binom{j}{i} \rho(|j\mu-i\lambda|) = I_1 + I_2$$

where I_1 is the sum over indices such that $|j\mu - i\lambda| < \delta$, while I_2 is the sum over indices satisfying $|j\mu - i\lambda| \geqq \delta$. Clearly $I_1 \leqq n\rho(\delta)$, while

$$
\begin{aligned}
I_2 &\leqq \rho(T) \sum_{j=0}^{n-1} \sum_{i=0}^{j} \beta^{j-i}\alpha^i \binom{j}{i} \frac{(j\mu-i\lambda)^2}{\delta^2} \\
&= \frac{\rho(T)}{\delta^2} n(n-1)(\lambda\mu-\mu^2) \leqq \frac{\rho(T)n^2}{\delta^2}\mu(\lambda-\mu)
\end{aligned}
$$

since $(j\mu - i\lambda)^2/\delta^2 \geqq 1$ for indices corresponding to I_2.
Therefore,

$$
\begin{aligned}
&\sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \beta_1^{j-i}\alpha_1^i \binom{j}{i} b_{m-i\,n-j} \\
&\leqq K(1-\mu\omega)^{-n} n\mu[\rho(|n\mu-m\lambda|) + \rho(\delta) + \frac{\rho(T)}{\delta^2} n\mu(\lambda-\mu)]
\end{aligned} \tag{2.24}
$$

for all $\delta > 0$. Combining (2.21) and (2.24) we obtain the existence of a K such that

$$
\begin{aligned}
a_{m,n} &\leqq K\{[(n\mu-m\lambda)^2 + n\mu(\lambda-\mu)]^{\frac{1}{2}} \\
&+ [(n\mu-m\lambda)^2 + m\lambda(\lambda-\mu)]^{\frac{1}{2}} + n\mu\rho(|n\mu-m\lambda|) + n\mu\rho(\delta) \\
&+ \frac{n^2\mu^2}{\delta^2}(\lambda-\mu)\}
\end{aligned} \tag{2.25}
$$

where a review of the estimates shows K can be taken to depend only on $\|x\|$, $M(x)$, $\rho(T)$, ω, λ_0 and T. We can choose, e.g., $\delta^2 = \sqrt{(\lambda - \mu)}$ and read off from (2.25) that $a_{m,n}$, as a function of m, n, μ and λ, tends to zero as $|n\mu - m\lambda| \to 0$ and $n, m \to \infty$, subject to $n, m \geqq 1$, $0 < n\mu$, $m\lambda \leqq T - s$, and that this limit is uniform in s. It follows that

$$U(s+\tau, s)x = \lim_{m\to\infty} \prod_{i=1}^{m} J_{\lambda m}(s + i\lambda_m)x, \quad x \in \hat{D}$$

exists if $\{\lambda_m\}$ is a sequence such that $0 \leqq m\lambda_m \leqq T - s$, $m\lambda_m \to \tau$ as $m \to \infty$ and that the limit is uniform in s and independent of $\{\lambda_m\}$. Since $\prod_{i=1}^{n} J_{(t-s)/n}(s + i(t-s)/n)$ has $(1 - \omega(t-s)/n)^{-n}$ as a Lipschitz constant on $\bar{D}$, and $(1 - \omega(t-s)/n)^{-n} \to e^{\omega(t-s)}$ as $n \to \infty$, it follows that the limit defining $U(t,s)$ exists for every $x \in \bar{D}$ uniformly on $0 \leqq s < t \leqq T$, and that

$$\| U(t,s)x - U(t,s)y \| \leqq e^{\omega(t-s)} \| x - y \|$$

for $x, y \in \bar{D}$. This concludes the proof of the existence of $U(t,s)$ and the estimate (2.6). It remains to verify the properties of an evolution operator.

Let $0 \leqq r \leqq T$, $\tau, t > 0$ and $r + \tau + t \leqq T$. Choose a sequence $\{k(n)\}$ of integers such that $k(n)t/n \leqq \tau$ and $k(n)t/n \to \tau$ as $n \to \infty$. Then

$$\prod_{i=1}^{k(n)} J_{t/n}(r + t + it/n) \prod_{i=1}^{n} J_{t/n}(r + it/n)x = \prod_{i=1}^{(n+k(n))} J_{t/n}(r + it/n)x.$$

Now $(n + k(n))(t/n) \to t + \tau$, and the uniform Lipschitz continuity and strong convergence of the operators involved allows us to take the limit as $n \to \infty$ and conclude

$$U(r + t + \tau, r + t)U(r + t, r)x = U(r + t + \tau, r)x$$

for $x \in \bar{D}$ and r, t, τ as above, thus verifying the defining identity for evolution operators.

The continuity of $U(t,s)$ in (t,s) is established in the next propositions.

PROPOSITION 2.1. *Let $x \in \hat{D}$ and either* (C.1) *or* (C.2) *hold. Let ρ be given by* (2.16) *and not be identically zero. Then there is a constant K such that*

$$\| U(t,s)x - U(\tau,s)x \| \leqq K\rho(|t - \tau|) \tag{2.26}$$

for $0 \leqq s \leqq t, \tau \leqq T$.

PROOF. Let $\lambda = (\tau/m)$, $\mu = (t/n)$ in (2.25) and take the limit as $n, m \to \infty$. This yields

$$\| U(t,s)x - U(\tau,s)x \| \leqq K(|t-\tau| + \rho(|t-\tau|) + \rho(\delta))$$

for a suitable constant K and any $\delta > 0$. Letting δ tend to zero we find

$$\| U(t,s)x - U(\tau,s)x \| \leqq K(|t-\tau| + \rho(|t-\tau|)).$$

Since ρ is continuous, subadditive and $\rho(0) = 0$, there is a constant c such that $r \leqq c\,\rho(r)$ unless $\rho \equiv 0$. The result follows at once.

REMARK. Proposition 2.1 shows that $U(t,s)x$ inherits exactly the continuity in t assumed for the $J_\lambda(t)$ if $x \in \hat{D}$.

PROPOSITION 2.2. *Let $x \in \hat{D}$ and either* (C.1) *or* (C.2) *hold. Then there is a constant K such that*

$$\| U(s+\tau,s)x - U(r+\tau,r)x \| \leqq K\rho(|r-s|) \tag{2.27}$$

if $0 \leqq \tau$ and $0 \leqq s, r, s+\tau, r+\tau \leqq T$.

PROOF. Let $a_k = \| P_{\lambda,k}(s)x - P_{\lambda,k}(r)x \|$. Then

$$\begin{aligned} a_k &= \| J_\lambda(s+k\lambda)P_{\lambda,k-1}(s)x - J_\lambda(r+k\lambda)P_{\lambda,k-1}(r)x \| \\ &\leqq \| J_\lambda(s+k\lambda)P_{\lambda,k-1}(s)x - J_\lambda(r+k\lambda)P_{\lambda,k-1}(s)x \| \\ &\quad + \| J_\lambda(r+k\lambda)P_{\lambda,k-1}(s)x - J_\lambda(r+k\lambda)P_{\lambda,k-1}(r)x \| \\ &\leqq \lambda C\rho(|s-r|) + (1-\lambda\omega)^{-1}a_{k-1} \end{aligned}$$

for a suitable constant C, where the first term was estimated by means of either (2.1) or (2.2) in conjunction with Lemma 2.3. Since $a_0 = 0$ this implies

$$a_n \leqq \lambda C\rho(|s-r|)\left(\sum_{i=0}^{n-1}(1-\lambda\omega)^{-i}\right) \leqq \lambda n\, K\,\rho(|s-r|)$$

where $K = C$ for $\omega \leqq 0$ and $K = C\exp(T\omega/(1-\lambda_0\omega))$ if $\omega > 0$. Substituting $\lambda = \tau/n$ and letting $n \to \infty$, we obtain the result.

COROLLARY 2.1. *Let* (C.1) *or* (C.2) *hold and $x \in \bar{D}$. Then $U(t,s)x$ is continuous in (t,s) on the triangle $0 \leqq s \leqq t \leqq T$.*

PROOF. In view of (2.6) it is enough to establish the result for $x \in \hat{D}$. If $x \in \hat{D}$, then $f(\tau,s) = U(s+\tau,s)x$ is continuous in τ uniformly in s by (2.26) and continuous in s uniformly in τ by (2.27), and the Corollary follows at once.

Corollary 2.1 completes the proof of Theorem 2.1. We conclude this section with some remarks concerning the case (C.2) and the rate at which $P_{\lambda,k}$ converges to $U(t,s)$. First we note that Proposition 2.1 can be strengthened if (C.2) holds.

PROPOSITION 2.3. *Let* (C.2) *hold and* $x \in \hat{D}$. *Then there exists a constant* C *such that*

$$\| U(t,s)x - U(\tau,s)x \| \leqq C|t - \tau|. \tag{2.28}$$

for $0 \leqq s \leqq t, \tau \leqq T$.

PROOF. Assume $t > \tau$ and let n and m be the greatest integers in t/λ and τ/λ respectively, where $0 < \lambda < \lambda_0$. According to Lemma 2.2 we have

$$\| P_{\lambda,n}(s)x - P_{\lambda,m}(s)x \| = \left\| \prod_{i=m+1}^{n} J_\lambda(s + i\lambda)P_{\lambda,m} - P_{\lambda,m} \right\| \leqq \lambda(n - m)KM(P_{\lambda m}).$$

Lemma 2.3 provides a bound on $M(P_{\lambda,m})$, so the result follows upon letting λ tend to zero.

PROPOSITION 2.4. *Let* (C.2) *hold. Then* $U(t,s): \hat{D} \to \hat{D}$ *for* $0 \leqq s \leqq t \leqq T$.

PROOF. $U(t,s)x = \lim_{n\to\infty} P_{(t-s)/n,n}(s)x$ and Lemma 2.3 provides a bound on $M(P_{(t-s)/n,n}(s)x)$ for $x \in \hat{D}$. The result follows at once from Lemma 1.4.

REMARK 2.1. The requirement that f be of bounded variation in (C.2) is used only to prove Lemma 2.3. All our results remain true and have the same proofs if one assumes Lemma 2.3, or any condition which implies it, and only continuity of f. If we are willing to give up Proposition 2.3, (C.2) could be changed to require only that for each $x \in \hat{D}$ there is a continuous f_x such that

$$\| J_\lambda(t)P_{\lambda,k}(s)x - J_\lambda(\tau)P_{\lambda,k}(s)x \| \leqq \lambda \| f_x(t) - f_x(\tau) \|$$

for suitable λ, t, τ, s, k. In particular, our assumptions can be localized.

PROPOSITION 2.5. *Let the assumptions of Theorem* 2.1 *hold and* $x \in \hat{D}$. *Then*

$$\left\| U(t,s)x - \prod_{i=1}^{m} J_{(t-s)/m}\left(s + i\frac{(t-s)}{m}\right)x \right\| \leqq K(t-s)\left(\frac{1}{\sqrt{m}} + \rho((t-s)/m^{\frac{1}{4}})\right). \tag{2.29}$$

where K *depends only on* $\| x \|$, $M(x)$, $\rho(T)$, ω, λ_0 *and* T.

PROOF. Substitute $\mu = (t-s)/m$, $\lambda = (t-s)/n$ in (2.25) and let $n \to \infty$. Then take $\delta^2 = (t-s)^2/\sqrt{m}$ to obtain the result.

REMARK 2.2. The estimate (2.29) can be sharpened in the special case of Hölder continuous f, i.e. $\rho(r) \leqq$ const. r^α, $0 < \alpha < 1$, to

$$\left\| U(t,s)x - \prod_{i=1}^{m} J_{(t-s)/m}\left(s + i\,\frac{(t-s)}{m}\right)x \right\| \leqq K(t-s)\left(\frac{1}{\sqrt{m}} + \frac{(t-s)^{\alpha}}{m^{(\alpha/2)}}\right).$$

3. The evolution equation

In this section we consider the Cauchy problem

$$\text{(3.1)} \qquad \begin{cases} \dfrac{du}{dt}(t) + A(t)u(t) \ni 0 \\ u(s) = x. \end{cases}$$

A function $u\colon [s,T] \to X$ is called a strong solution of (3.1) on $[s,T]$ if:

(i) u is continuous on $[s,T]$ and $u(s) = x$,

(ii) u is absolutely continuous on compact subsets of (s,T),

(iii) u is differentiable a.e. on (s,T) and satisfies (3.1) a.e.

The evolution operator $U(t,s)$ we constructed in Section 2 is intimately related to the solutions of (3.1). We will prove that if (3.1) has a strong solution $u(t)$, then $u(t) = U(t,s)x$. Furthermore, we will give various conditions under which $U(t,s)x$ is a strong solution of (3.1) provided it is differentiable a.e. Finally we obtain an existence theorem for strong solutions of (3.1) by noting that in certain cases $U(t,s)x$ is differentiable a.e. The conditions (A.1)–(A.3) of Section 2 are assumed to hold throughout this section.

THEOREM 3.1. *Let $u(t)$ be a strong solution of the initial value problem* (3.1) *on $[s,T]$. Let (C.1) or (C.2) hold. Then $U(t,s)x = u(t)$ for $0 \leqq s \leqq t \leqq T$.*

PROOF. Using the definition of a strong solution of (3.1) and the continuity of $U(t,s)x$, we easily reduce to the case where u is absolutely continuous on $[s,T]$ and $x \in \hat{D}$. For simplicity, we also assume $s = 0$. For each $\varepsilon > 0$ we define $u_\varepsilon\colon [0,T] \to X$ by

$$u_\varepsilon(t) = \prod_{i=0}^{[t/\varepsilon]} J_\varepsilon(i\varepsilon)x$$

where $[t/\varepsilon]$ is the greatest integer in t/ε. Then $u_\varepsilon(t)$ is a step function which solves the approximate problem

$$\begin{cases} 0 \in \dfrac{u_\varepsilon(t) - u_\varepsilon(t-\varepsilon)}{\varepsilon} + A([t/\varepsilon]\varepsilon)u_\varepsilon(t) \quad \text{for} \quad t \geqq 0 \\ u_\varepsilon(t) = x \quad \text{for} \quad 0 > t. \end{cases}$$

More compactly, we have

$$\text{(3.2)} \qquad \begin{cases} u_\varepsilon(t) = J_\varepsilon([t/\varepsilon]\varepsilon)\, u_\varepsilon(t-\varepsilon) & \text{for} \quad t \geqq 0 \\ u_\varepsilon(t) = x & \text{for} \quad 0 > t. \end{cases}$$

From the definition of $u_\varepsilon(t)$ and the proof of Theorem 2.1 it follows

$$\text{(3.3)} \qquad \lim_{\varepsilon \downarrow 0} \| u_\varepsilon(t) - U(t,0)x \| = 0$$

uniformly for $0 \leqq t \leqq T$. Next we set $u(t) = x$ for $0 > t$ and

$$g_\varepsilon(t) = -u'(t) + \frac{u(t) - u(t-\varepsilon)}{\varepsilon} \quad \text{a.e.}$$

Since u is a strong solution of (3.1)

$$\text{(3.4)} \qquad u(t) = J_\varepsilon(t)(u(t-\varepsilon) + \varepsilon g_\varepsilon(t))$$

a.e. on $[0, T]$ and, since u is absolutely continuous on $[0, T]$,

$$\text{(3.5)} \qquad \lim_{\varepsilon \downarrow 0} \int_0^T \| g_\varepsilon(s) \| \, ds = 0.$$

From (3.2) and (3.3) we have

$$\begin{aligned} \| u_\varepsilon(t) - u(t) \| &= \| J_\varepsilon([t/\varepsilon]\varepsilon) u_\varepsilon(t-\varepsilon) - J_\varepsilon(t)(u(t-\varepsilon) + \varepsilon g_\varepsilon(t)) \| \\ &\leqq \| J_\varepsilon(t) u_\varepsilon(t-\varepsilon) - J_\varepsilon(t)(u(t-\varepsilon)) \| + \varepsilon(1-\varepsilon\omega)^{-1} \| g_\varepsilon(t) \| \\ &\quad + \| J_\varepsilon(t) u_\varepsilon(t-\varepsilon) - J_\varepsilon([t/\varepsilon]\varepsilon) u_\varepsilon(t-\varepsilon) \| \\ &\leqq (1-\varepsilon\omega)^{-1}(\| u_\varepsilon(t-\varepsilon) - u(t-\varepsilon) \| + \varepsilon \| g_\varepsilon(t) \|) + \varepsilon K \rho(t - [t/\varepsilon]\varepsilon) \end{aligned}$$

a.e. on $[0, T]$. Here we used either (C.1) or (C.2) in conjunction with Lemma 2. Integrating this last inequality over $[0, t]$ and rearranging we find

$$\begin{aligned} &1/\varepsilon \int_{t-\varepsilon}^t \| u_\varepsilon(s) - u(s) \| \, ds \\ &\leqq \frac{\omega}{1-\varepsilon\omega} \int_0^t \| u_\varepsilon(s-\varepsilon) - u(s-\varepsilon) \| \, ds + (1-\varepsilon\omega)^{-1} \int_0^t \| g_\varepsilon(s) \| \, ds + K t \rho(\varepsilon). \end{aligned}$$

Letting $\varepsilon \to 0$ and using (3.3) and (3.4) gives

$$\| U(t,0)x - u(t) \| \leqq \omega \int_0^t \| U(s,0)x - u(s) \| \, ds$$

for $0 \leqq t \leqq T$, which implies $U(t,0)x = u(t)$ for $0 \leqq t \leqq T$. The proof is complete.

Next we want to relate $(d/dt)\, U(t,s)x$, whenever it exists, to $A(t)U(t,s)x$. This is done under several different hypotheses. We begin with a rather technical result.

THEOREM 3.2. *Let $A(t)$ satisfy the conditions of Theorem 2.1. Moreover, let:*
(i) *$A(t)$ be a closed subset of $X \times X$ for $0 \leqq t \leqq T$.*
and

(ii) *For every $t_0, 0 \leqq t_0 < T$, and $[x_0, y_0] \in A(t_0)$ there exists a continuous function $y(t)$ on some interval $[t_0, t_0 + \delta]$, $\delta > 0$, such that $y(t_0) = y_0$ and $y(t) \in A(t)x_0$ for $t_0 \leqq t \leqq t_0 + \delta$.*

If $x \in \bar{D}$ and the (two-sided) derivative $d/dt\ U(t,s)x$ exists at some point (t,s), $0 \leqq s < t < T$, then $U(t,s)x \in D(A(t))$ and

$$0 \in \frac{d}{dt} U(t,s)x + A(t)U(t,s)x.$$

PROOF. Let $F(x) = \{x^* \in X^* : (x, x^*) = \| x \|^2 = \| x^* \|^2\}$, where (x, x^*) denotes the value of $x^* \in X^*$ at $x \in X$, and

$$\langle y, x \rangle_s = \sup \{(y, x^*) : x^* \in F(x)\} = \max \{(y, x^*) : x^* \in F(x)\}.$$

It is known that

$$2\langle y, x \rangle_s = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(\| x + \varepsilon y \|^2 - \| x \|^2) = \inf_{\varepsilon > 0} \frac{1}{\varepsilon}(\| x + \varepsilon y \|^2 - \| x \|^2)$$

from which it follows that the map $[y, x] \to \langle y, x \rangle_s$ is upper semicontinuous in the pair $[y, x]$ (and Lipschitz continuous in y for fixed x). Another simple proof is given in [10]. It is also known that $A \in \mathscr{A}(\omega)$ is equivalent to the condition that $\langle y_1 - y_2, x_1 - x_2 \rangle_s \geqq -\omega \| x_1 - x_2 \|^2$ for each pair $[x_i, y_i] \in A$. See [14] for a proof in the case $\omega = 0$. Now, let $0 \leqq t_0 < T$, $[x_0, y_0] \in A(t_0)$ and $[x_0, y(t)] \in A(t)$ such that $y(t_0) = y_0$ and $y(t)$ is continuonus. By definition of J_λ we have $\lambda^{-1}(x - J_\lambda(t)x) \in A(t)J_\lambda(t)x$, so

$$\lambda^{-1}(P_{\lambda,k-1}(t_0)z - P_{\lambda,k}(t_0)z) \in A(t_0 + k\lambda)P_{\lambda,k}(t_0)z$$

for every $z \in \bar{D}$ and $0 < \lambda < \lambda_0$. Since $A(t_0 + k\lambda) \in \mathscr{A}(\omega)$ there exists $\eta^* \in F(x_0 - P_{\lambda,k}(t_0)z)$ such that:

$$(y(t_0 + k\lambda) - \lambda^{-1}(P_{\lambda,k-1} - P_{\lambda,k}), \eta^*) \geqq -\omega \| x_0 - P_{\lambda,k} \|^2$$

and since $\eta^* \in F(x_0 - P_{\lambda,k})$, this implies

$$\begin{aligned}
&\langle y(t_0 + k\lambda) + \omega(x_0 - P_{\lambda,k}), x_0 - P_{\lambda,k} \rangle_s \geqq \lambda^{-1}(\| x_0 - P_{\lambda,k} \|^2 \\
&- (x_0 - P_{\lambda,k-1}, \eta^*)) \geqq \lambda^{-1}(\| x_0 - P_{\lambda,k} \|^2 - \| x_0 - P_{\lambda,k-1} \| \, \| x_0 - P_{\lambda,k} \|) \\
&\geqq (\tfrac{1}{2}\lambda)(\| x_0 - P_{\lambda,k} \|^2 - \| x_0 - P_{\lambda,k-1} \|^2)
\end{aligned}$$

for $k = 1, 2, \cdots, [t/\lambda]$. Summing over k, $1 \leqq k \leqq [t/\lambda]$, we obtain

$$2\lambda \sum_{k=1}^{[t/\lambda]} \langle y(t_0 + k\lambda) + \omega(x_0 - P_{k,\lambda}), x_0 - P_{k,\lambda}\rangle_s \geqq \| x_0 - P_{\lambda,[t/\lambda]} \|^2 - \| x_0 - z \|^2. \tag{3.6}$$

Set $f_\lambda(\tau) = \langle y(t_0 + k\lambda) + \omega(x_0 - P_{k,\lambda}),\ x_0 - P_{k,\lambda}\rangle_s$ for $k\lambda \leqq \tau < (k+1)\lambda$. Then (3.6) may be restated as

$$2 \int_0^{[t/\lambda]\lambda} f_\lambda(\tau)\, d\tau \geqq \| x_0 - P_{\lambda,[t/\lambda]} \|^2 - \| x_0 - z \|^2. \tag{3.7}$$

Moreover, the upper semicontinuity of $\langle\ ,\ \rangle_s$ and $P_{\lambda,[\tau/\lambda]} \to U(t_0 + \tau, t_0)z$ as $\lambda \downarrow 0$ imply

$$\limsup_{\lambda \downarrow 0} f_\lambda(\tau) \leqq \langle y(t_0 + \tau) + \omega(x_0 - U(t_0 + \tau, t_0)z), x_0 - U(\tau + t_0, t_0)z\rangle_s$$

where the right-hand side is upper semicontinuous and therefore integrable. Thus, letting $\lambda \downarrow 0$ in (3.7) and dividing by t we find

$$\begin{aligned} &\frac{1}{t}\int_0^t \langle y(t_0 + \tau) + \omega(x_0 - U(t_0 + \tau, t_0)z), x_0 - U(t_0 + \tau, t_0)z\rangle_s d\tau \\ &\geqq \frac{1}{2t}(\| x_0 - U(t + t_0, t_0)z \|^2 - \| x_0 - z \|^2) \geqq \left(\frac{z - U(t + t_0, t_0)z}{t}, \xi^*\right) \end{aligned} \tag{3.8}$$

for all $\xi^* \in F(x_0 - z)$, where the second inequality is obvious. Now assume the right derivative

$$D_+ U(t,s)x \big|_{t=t_0} = D_+ U(t,t_0)U(t_0,s)x \big|_{t=t_0}$$

exists. Let $z = U(t_0,s)x$ in (3.8) and let $t \downarrow 0$ to obtain

$$\langle y_0 + \omega(x_0 - U(t_0,s)x), x_0 - U(t_0,s)x\rangle_s \geqq (-D_+ U(t_0,s)x, \xi^*)$$

for every $\xi^* \in F(x_0 - U(t_0,s)x)$. In view of the arbitrary choice of $[x_0, y_0] \in A(t_0)$, it follows that

$$A(t_0) \cup \{[U(t_0,s)x, -D_+ U(t_0,s)x]\} \in \mathscr{A}(\omega). \tag{3.9}$$

If the two-sided derivative $d/dt\, U(t,s)x \big|_{t=t_0} = y$ exists, we have

$$U(t_0 - h, s)x = U(t_0,s)x - hy + o(h) \tag{3.10}$$

as $h \to 0$. Since $y = D_+ U(t_0,s)x$, (3.9) and (3.10) imply

$$x_h = J_h(t_0)(U(t_0 - h, s)x) = U(t_0,s)x + o(h)$$

and

$$y_h = \frac{U(t_0 - h, s)x - J_h(t_0)\, U(t_0 - h, s)x}{h} = -y + o(1)$$

as $h \downarrow 0$. But $[x_h, y_h] \in A(t_0)$, $x_h \to U(t_0,s)x, y_h \to -y$, and the closedness of $A(t_0)$ then imply $[U(t_0,s), -y] \in A(t_0)$ and the proof is complete.

REMARK 3.1. The inequality (3.8) is a generalization of a result of [2] for X^* uniformly convex and A independent of t. This was extended to general X in [10]. Here we have used the idea of proof of [18].

REMARK 3.2. It is clear that the proof of Theorem 3.2 goes through if one only requires that

$$\lim_{t\downarrow 0} \frac{1}{t}\int_0^t \| y(s+\tau) - y_0 \| \, d\tau = 0$$

rather than condition (ii) of the theorem.

The conclusion of Theorem 3.2 can be established without requiring the condition (ii) in the following case:

THEOREM 3.3. *Let each $A(t)$ be a closed subset of $X \times X$ and $x \in \bar{D}$. If* (C.1) *holds or $x \in \hat{D}$ and* (C.2) *holds, then the assertions of Theorem 3.2 concerning $U(t,s)x$ remain true.*

PROOF. Let $S_\tau(t)$ be the semigroup on $\bar{D}$ generated by $A(\tau)$, i.e.,

$$S_\tau(t) = \lim_{n\to\infty} \left(I + \frac{t}{n}A(\tau)\right)^{-n}.$$

The existence of $S_\tau(t)$ follows at once from Theorem 2.1 and was first proved in [10]. We estimate $\| U(\tau+h,\tau)x - S_\tau(h)x \|$ in the next lemma.

LEMMA 3.1. *If $x \in \bar{D}$ and* (C.1) *holds or $x \in \hat{D}$ and* (C.2) *holds, then*

$$\lim_{h\downarrow 0} \frac{\| S_\tau(h)x - U(\tau+h,\tau)x \|}{h} = 0$$

for $0 \leqq \tau < T$.

PROOF. Let $a_k = \| P_{\lambda,k}(\tau)x - J_\lambda^k(\tau)\, x \|$. Then

$$\begin{aligned} a_k &= \| J_\lambda(\tau + k\lambda)P_{\lambda,k-1}x - J_\lambda^k(\tau)\, x \| \leqq \| J_\lambda(\tau+k\lambda)P_{\lambda,k-1} - J_\lambda(\tau)P_{\lambda,k-1} \| \\ &+ \| J_\lambda(\tau)P_{\lambda,k-1} - J_\lambda^k(\tau)\, x \| \leqq \lambda K\rho(k\lambda) + (1-\lambda\omega)^{-1}a_{k-1}. \end{aligned} \tag{3.12}$$

Here we used (C.1) or Lemma 2.3 together with (C.2). Solving (3.12) gives, since $a_0 = 0$,

$$a_n \leqq K(1-\lambda\omega)^{-n}\lambda \sum_{k=1}^{n} \rho(\lambda k) \leqq K(1-\lambda\omega)^{-n}\lambda n\rho(n\lambda).$$

Let $n = [h/\lambda]$ as $\lambda \downarrow 0$ to obtain

$$\| U(\tau + h, \tau)x - S_\tau(h)x \| \leqq K_1 h \rho(h).$$

The proof of the lemma is complete. We return to the proof of the theorem. Note that the "family" of operators $A^1(t) = A(t_0)$ for $0 \leqq t \leqq T$ satisfies the assumptions of Theorem 3.2. According to the proof of that theorem, $A(t_0) \cup \{[S_{t_0}(t)x, -D_+S_{t_0}(t)x]\}$ is in $\mathscr{A}(\omega)$ if the right derivative exists. By Lemma 3.1 we have $D_+S_{t_0}(t)z\,|_{t=0} = D_+U(t,t_0)z\,|_{t=t_0}$ provided $z \in \bar{D}$ and (C.1) holds or $z \in \hat{D}$ and (C.2) holds. Set $z = U(t_0, s)x$ (and recall Prop. 2.4), so that we have

$$\begin{aligned} &A(t_0) \cup \{[S_{t_0}(0)z, -D_+S_{t_0}(t)z\,|_{t=0}]\} \\ &\quad = A(t_0) \cup \{[U(t_0,s)x, -D_+U(t_0,s)x]\} \in \mathscr{A}(\omega). \end{aligned}$$

The last argument in the proof of Theorem 3.3 can now be repeated here, and the proof is complete.

We conclude this section with an existence theorem providing strong solutions of (3.1) and some remarks.

THEOREM 3.4. *Let X be a reflexive Banach space and $A(t)$ be closed for each t, $0 \leqq t \leqq T$. Let* (C.2) *hold. Then for every $x \in \hat{D}$ and $0 \leqq s < T$ the initial value problem* (3.1) *has a unique solution $u(t)$ given by*

$$u(t) = U(t,s)x = \lim_{n \to \infty} \prod_{k=1}^{n} \left(I + \frac{t-s}{n} A\left(s + k\frac{t-s}{n}\right)\right)^{-1} x.$$

Moreover, $u(t) \in \hat{D}$ for $s \leqq t \leqq T$.

PROOF. Proposition 2.3 shows that $U(t,s)x$ is Lipschitz continuous in t and is therefore absolutely continuous. Since X is reflexive, $U(t,s)x$ is differentiable a.e. on $[s,T]$. It follows immediately from Theorem 3.3 that $U(t,s)x$ is a strong solution of (3.1) and uniqueness is evident from Theorem 3.1. The proof is complete.

REMARK 3.3. Theorem 3.4 and its proof remain valid if the condition that X be reflexive is weakened to the condition that X-valued Lipschitz continuous functions of t are differentiable a.e. For example, this is the case for $X = l_1$.

Theorem 3.4 is a generalization of the results of Kato [14] for the case of single-valued $A(t)$, uniformly convex X^*, and $\|f(t) - f(\tau)\| = |t - \tau|$ and $\omega = 0$. We relate our (C.2) to Kato's continuity assumption below.

LEMMA 3.2. *Let $A(t)$ be single-valued and satisfy* (A.1) *and* (A.3) *and $D(A(t)) = D$ be independent of t. If*

$$(3.13) \qquad \| A(t)x - A(\tau)x \| \leqq |t - \tau| L(\|x\|)(1 + \|A(t)x\|)$$

for $x \in D$ *and* $0 \leqq \tau, t \leqq T$, *then*

$$(3.14) \qquad \| J_\lambda(t)x - J_\lambda(\tau)x \| \leqq \lambda |t - \tau| L_1(\|x\|)(1 + |A(t)x|)$$

for $x \in \bar{D}$, $0 < \lambda < 1$, $\lambda\omega < \frac{1}{2}$, *and* $0 \leqq t, \tau \leqq T$, *where* $L_1(r) = 4L(K(r+1))$ *for a suitable* K. *Moreover, if* X^* *is uniformly convex, and* $R(I + \lambda A(t)) = X$ *for* $0 < \lambda < \lambda_0$, $0 \leqq t \leqq T$, *then* (3.14) *implies* (3.13) *with* $L = L_1$.

PROOF. Assume $A(t)$ is single-valued and that (3.13) holds. Then, if $x \in \bar{D}$,

$$\begin{aligned}
\| J_\lambda(\tau)x - J_\lambda(t)x \| &= \| J_\lambda(t)(I + \lambda A(t))J_\lambda(\tau)x - J_\lambda(t)(I + \lambda A(\tau))J_\lambda(\tau)x \| \\
&\leqq (1 - \lambda\omega)^{-1} \| (I + \lambda A(t))J_\lambda(\tau)x - (I + \lambda A(\tau))J_\lambda(\tau)x \| \\
&= \lambda(1 - \lambda\omega)^{-1} \| A(t)J_\lambda(\tau)x - A(\tau)J_\lambda(\tau)x \| \\
(3.15) \qquad &\leqq 2\lambda |t - \tau| L(\|J_\lambda(\tau)x\|)(1 + \|A(\tau)J_\lambda(\tau)x\|) \\
&\leqq 2\lambda |t - \tau| L(\|J_\lambda(\tau)x\|)(1 + (1 - \lambda\omega)^{-1}|A(\tau)x|). \\
&\leqq 4\lambda |t - \tau| L(\|J_\lambda(\tau)x\|)(1 + |A(\tau)x|)
\end{aligned}$$

Now let $y \in D$ be fixed. Then

$$\begin{aligned}
\| J_\lambda(\tau)x - y \| &= \| J_\lambda(\tau)x - J_\lambda(\tau)(I + \lambda A(\tau))y \| \\
&\leqq (1 - \lambda\omega)^{-1} \| x - (y + \lambda A(\tau)y) \| \leqq 2(\|x\| + \|y\| + \lambda \|A(\tau)y\|).
\end{aligned}$$

Since $\|A(\tau)y\|$ is bounded for $0 \leqq \tau \leqq T$ by (3.13), $\|J_\lambda(\tau)x\| \leqq K(\|x\| + 1)$ for some K, $0 < \lambda < 1$. Thus (3.13) implies (3.14). On the other hand, (3.14) implies

$$\begin{aligned}
\| A_\lambda(t)x - A_\lambda(\tau)x \| &= \| \lambda^{-1}(x - J_\lambda(t)x) - \lambda^{-1}(x - J_\lambda(\tau)x) \| \\
&\leqq |t - \tau| L_1(\|x\|)(1 + |A(t)x|).
\end{aligned}$$

If X^* is uniformly convex, $A(t) \in \mathscr{A}(\omega)$, $A(t)$ is single-valued and $R(I + \lambda A(t)) = X$ for small $\lambda > 0$, and $x \in D(A)$, then $A_\lambda(t)x$ converges weakly to $A(t)x$ as $\lambda \downarrow 0$. Moreover $\|A(t)x\| = |A(t)x|$ for $x \in D(A)$. Hence, letting $\lambda \downarrow 0$ above,

$$\| A(t)x - A(\tau)x \| \leqq \liminf_{\lambda \downarrow 0} \| A_\lambda(t)x - A_\lambda(\tau)x \| \leqq |t - \tau| L_1(\|x\|)(1 + \|A(t)x\|).$$

The proof is complete.

REMARK 3.4. If X and X^* are uniformly convex, Theorem 3.4 can be strengthened. In this case, it follows from the assumptions of Theorem 3.4 that $D(A(t)) = \hat{D}$ for $0 \leqq t \leqq T$, and if $x \in \hat{D}$ then the set $A(t)x$ has a unique element

$A(t)^0x$ such that $\| A(t)^0x \| = | A(t)x |$. The solution $u(t)$ of the initial-value problem (3.1) is everywhere differentiable from the right and

$$\begin{cases} D_+u(t) + A(t)^0u(t) = 0 & s \leqq t \leqq T \\ u(s) = x. \end{cases}$$

See also [17].

4. Approximation and continuous dependence of evolution operators

For each β, $0 \leqq \beta < 1$, let $A^\beta(t)$ be a family of $\mathscr{A}(\omega)$ sets satisfying the assumptions of Theorem 2.1 and let $U^\beta(t,s)$ be the corresponding evolution operators. Suppose $A^\beta(t) \to A^0(t)$ (in some sense) as $\beta \to 0$. One expects that $U^\beta(t,s)$ will converge to $U^0(t,s)$. Our first result, Theorem 4.1, shows this is indeed the case under certain weak conditions. This theorem extends a result of Brezis and Pazy [5] in the t-independent case. Using Theorem 4.1, we show that an evolution operator provided by Theorem 2.1 can be written as the limit of C^1 evolution operators $U^\beta(t,s)$ obtained by Yosida's type of approximation. We drop the index β if $\beta = 0$ below, e.g., $A^0(t) = A(t)$, $U^0(t,s) = U(t,s)$ etc.

THEOREM 4.1. *Let $A^\beta(t)$ satisfy the assumptions of Theorem 2.1 uniformly in β, $0 \leqq \beta < 1$ (i.e., we may take the same ω, λ_0, T, f, L for each $A^\beta(t)$). Let $D_\beta = D(A^\beta(0))$, $J_\lambda^\beta(t) = (I + \lambda A^\beta(t))^{-1}$ and assume*

(4.1) $$\lim_{\beta \downarrow 0} J_\lambda^\beta(t)x = J_\lambda(t)x \text{ for } x \in Q,\ 0 < \lambda < \lambda_0,\ 0 \leqq t \leqq T$$

where $Q = (\bigcap_{1 > \beta \geqq 0} \bar{D}_\beta) \cap \hat{D}$, $\hat{D} = \hat{D}(A(0))$. Let $U^\beta(t,s)$ be the evolution operator on $\bar{D}_\beta$ associated with $A^\beta(t)$ in the sense of Theorem 2.1. Then

$$\lim_{\beta \downarrow 0} U^\beta(t,s)x = U(t,s)x$$

for $x \in \bar{Q}$, $0 \leqq s \leqq t \leqq T$, and the limit is uniform in $t \in [s,T]$.

PROOF. We assume, in the proof, that the operators satisfy (C.2). (The proof for (C.1) is the same and we could even allow a mixture of (C.1) and (C.2).) The idea of the proof is simple. For each β, let $P_{\mu,k}^\beta(s)x$ be defined as in (2.11). By Theorem 2.1, $U^\beta(t,s)x = \lim_{n\to\infty} P_{(t-s)/n,n}^\beta(s)x$ for each β, while $\lim_{\beta\downarrow 0} P_{\mu,n}^\beta(s)x = P_{\mu,n}(s)x$ follows at once from our assumptions. The argument below is thus to show that we may exchange the order of limits in the iterated expression $\lim_{n\to\infty} \lim_{\beta\downarrow 0} P_{(t-s)/n,n}^\beta(s)x$.

Since (C.2) is satisfied uniformly in β, the proof of Theorem 2.1 and Proposition 2.5 show

$$\lim_{n\to\infty} \| U^{\beta}(t,s)z - P^{\beta}_{(t-s)/n,n}(s)z \| = 0 \tag{4.2}$$

holds for $0 \leqq \beta < 1$, $z \in \bar{D}_{\beta}$. Moreover, given $R > 0$, (4.2) holds uniformly on any set of $[s,t,\beta,z]$ which satisfy

$$z \in \bar{D}_{\beta},\ \| z \| \leqq R,\ | A^{\beta}(0)z | \leqq R,\ 0 \leqq s \leqq t \leqq T. \tag{4.3}$$

We shall now show that if $x \in (\cap_{\beta>0}\bar{D}_{\beta}) \cap \hat{D}$ is fixed then there is a positive function $\beta_0(\lambda)$ such that

$$\{[s,t,\beta,z]: 0 \leqq s \leqq t \leqq T,\ 0 < \beta < \beta_0(\lambda),\ z = J^{\beta}_{\lambda}(0)x\}$$

satisfies (4.3) for some $R > 0$. Indeed, since $J^{\beta}_{\lambda}(0)x \to J_{\lambda}(0)x$ and $x \in \hat{D}$, there is an $R > 0$ and a function $\beta_0(\lambda) > 0$ such that if $0 < \lambda < \lambda_0$ and $0 < \beta < \beta_0(\lambda)$, then

$$\begin{aligned} |A^{\beta}(0)J^{\beta}_{\lambda}(0)x| &\leqq \| A^{\beta}_{\lambda}(0)x \| \leqq \| \lambda^{-1}(J^{\beta}_{\lambda}(0)x - x) \| \\ &\leqq (\| A_{\lambda}(0)x \| + 1) \leqq (1-\lambda_0\omega)^{-1}(|A(0)x| + 1) \leqq R \end{aligned} \tag{4.4}$$

and

$$\| J_{\lambda}(0)x \| \leqq R. \tag{4.5}$$

Now

$$\begin{aligned} \| U^{\beta}(t,s)x - U(t,s)x \| &\leqq \| U^{\beta}(t,s)x - U^{\beta}(t,s)J^{\beta}_{\lambda}(0)x \| \\ &+ \| U^{\beta}(t,s)J^{\beta}_{\lambda}(0)x - P^{\beta}_{(t-s)/n,n}(s)J^{\beta}_{\lambda}(0)x \| \\ &+ \| P^{\beta}_{(t-s)/n,n}(s)J^{\beta}_{\lambda}(0)x - P^{\beta}_{(t-s)/n,n}(s)x \| \\ &+ \| P^{\beta}_{(t-s)/n,n}(s)x - P_{(t-s)/n,n}(s)x \| + \| P_{(t-s)/n,n}(s)x - U(t,s)x \| \end{aligned}$$

Numbering the terms on the right above (1)–(5) in their order of appearance we have

$$(1) + (3) \leqq \left(e^{\omega(t-s)} + \left(1 - \left(\frac{t-s}{n}\right)\omega\right)^{-n}\right) \| J^{\beta}_{\lambda}(0)x - x \| \leqq 3e^{\omega T}\lambda R \tag{4.6}$$

for all n large enough and $0 < \beta < \beta_0(\lambda)$. Here we used (4.4). Next, given $\varepsilon > 0$ the uniformity of (4.2) as discussed there and (4.4), (4.5) show that there is an n_0 such that

$$(2) + (5) < \varepsilon/3 \text{ for } 0 < \beta < \beta_0(\lambda),\ n \geqq n_0. \tag{4.7}$$

The remaining term (4) tends to zero as $\beta \to 0$ for fixed n. Thus we proceed as follows; given $\varepsilon > 0$ choose λ such that $3e^{\omega t}\lambda R < \varepsilon/3$. Then pick n_0 such that (4.6) and (4.7) are satisfied for $n \geqq n_0$ and $0 < \beta < \beta_0(\lambda)$. For $n = n_0$, we then have $(1) + (2) + (3) + (5) \leqq (2/3)\varepsilon$ provided only that $0 < \beta < \beta_0(\lambda)$. Now there is a β_1, $0 < \beta_1 < \beta_0(\lambda)$ (depending on t, s, n_0) such that $(4) < \varepsilon/3$ if $n = n_0$ and $0 < \beta \leqq \beta_1$. Hence

$$\| U^\beta(t,s)x - U(t,s)x \| < \varepsilon$$

if $0 < \beta < \beta_1$. This proves $\lim_{\beta\to 0} U^\beta(t,s)x = U(t,s)x$ for fixed t,s, $0 \leqq s \leqq t \leqq T$ and $x \in Q$. To show the limit is uniform in $t \in [s,T]$, let $0 \leqq s \leqq t \leqq T$ and consider

$$\begin{aligned}\|U^\beta(t,s)x - U(t,s)x\| \leqq{}& \| U^\beta(t,s)x - U^\beta(t,s)J_\lambda^\beta(0)x \| \\ &+ \| U^\beta(t,s)J_\lambda^\beta(0)x - U^\beta(\tau,s)J_\lambda^\beta(0)x \| + \| U^\beta(\tau,s)J_\lambda^\beta(0)x \\ &- U^\beta(\tau,s)x\| + \|U^\beta(\tau,s)x - U(\tau,s)x \| + \|U(\tau,s)x - U(t,s)x \| \\ \leqq{}& 2e^{\omega t}\| J_\lambda^\beta(0)x - x \| + K\,\rho(|t-\tau|) + \| U^\beta(\tau,s)x - U(\tau,s)x \|\end{aligned}$$

where we used Proposition 2.1. We require the fact that K can be taken to be independent of λ, β provided $0<\beta<\beta_0(\lambda)$ so that (4.4) holds. Then, using (4.4) we have

$$\| U^\beta(t,s)x - U(t,s)x \| \leqq 2e^{\omega T}\lambda R + K\rho(|t-\tau|) + \| U^\beta(\tau,s)x - U(\tau,s)x \|$$

if $0 < \beta < \beta_0(\lambda)$. It follows at once that if $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} t_n = \tau$, then $\lim_{n\to\infty} U^{\beta_n}(t_n,s)x = U(\tau,s)x$, completing the proof for $x \in Q$. Since each $U^\beta(t,s)$ has $e^{\omega T}$ as a Lipschitz constant, the result for $x \in \bar{Q}$ follows from a limiting procedure, and the proof is complete.

We show next that if $A(t)$ satisfies the conditions of Theorem 2.1, with (A.3 strengthened to

(A.4) $\qquad R(I + \lambda A(t)) \supset \overline{\mathrm{conv}\, D}$ for $0 < \lambda < \lambda_0$, $0 \leqq t \leqq T$,

where conv D is the convex hull of D, then the operators $A^\beta(t) = A_\beta(t) = \beta^{-1}(I - J_\beta(t))$ satisfy the requirements of Theorem 4.1. We will see that $U^\beta(t,s)x$ is a continuously differentiable function of t for $x \in \bar{D}$ in this case, so it will follow that $U(t,s)$ is the limit of C^1 evolution operators.

LEMMA 4.1. *Let $A(t)$ satisfy* (A.1), (A.2), (A.4) *and satisfy* (C.2) (*respectively,* C.1). *Then $A_\beta(t) \in \mathscr{A}(\omega(1-\beta\omega)^{-1})$ for $0 < \beta < \lambda_0$. Moreover, the restriction of $A_\beta(t)$ to $C = \overline{\mathrm{conv}\, D}$ satisfies* (C.2) (*respectively* (C.1)) *uniformly in β, $0 < \beta < \lambda_0$, and*

$$R(I + \lambda A_\beta(t)|_C) \supset C$$

if $0 < \beta < \lambda_0$, $0 < \lambda$, $\lambda\omega(1 - \beta\omega)^{-1} < 1$.

PROOF. Lemma 1.2 (iii) shows $A_\beta(t) \in \mathscr{A}(\omega(1 - \beta\omega)^{-1})$. The assumption (A.4) implies $D(A_\beta(t)) \supseteq C$, so the restriction of $A_\beta(t)$ to C certainly satisfies (A.2). A slight change in the proof of Lemma 3.2 (and replacing $D(A(t))$ by $\hat{D}(A(t))$ in the assumptions), shows that if $A(t)$ satisfies (C.2) so does $A_\beta(t)$ with the same ρ and a slightly modified L. The case (C.1) is dealt with similarly. It remains to show that given $x \in C$ the equation $y + \lambda A_\beta(t)y = x$ has a solution $y \in C$ for $0 < \lambda$ and $\lambda\omega(1 - \beta\omega)^{-1} < 1$. This last equation is equivalent to $y = \psi_x(y)$, where

$$\psi_x(y) = \frac{\beta}{\lambda + \beta}x + \frac{\lambda}{\beta + \lambda}J_\beta(t)y.$$

However, $\psi_x : C \to C$ and ψ_x is a strict contraction if $\lambda\omega(1 - \beta\omega)^{-1} < 1$. The proof is complete.

LEMMA 4.2. *Let $A(t)$ satisfy the conditions of Lemma 4.1 and let $U^\beta(t,s)$ (respectively $U(t,s)$) be the evolution operator corresponding to $A_\beta(t)$ on C respectively $A(t)$ on $\bar{D}$). Then*

$$\lim_{\beta \downarrow 0} U^\beta(t,s)x = U(t,s)x$$

for every $x \in \bar{D}$ and the limit is uniform in t, for $s \leqq t \leqq T$.

PROOF. We verify the conditions of Theorem 4.1 with $A^\beta(t) = A_\beta(t)|_C$. In view of Lemma 4.1 we need only show $\lim_{\beta\downarrow 0} J_\lambda^\beta(t)x = J_\lambda(t)x$ for $x \in \bar{D}$, $0 < \lambda < \lambda_0$, where $J_\lambda^\beta(t) = (I + \lambda A_\beta(t))^{-1}x$. This is an immediate consequence of the identity

$$J_\lambda^\beta(t)x = J_{\lambda+\beta}(t)x + \beta A_{\lambda+\beta}(t)x \tag{4.8}$$

and the continuity of $J_\lambda(t)x$ (and hence of $A_\lambda(t)x$) in λ for $0 < \lambda < \lambda_0$. Both (4.8) and the continuity follow from the resolvent identity (1.1). We verify (4.8). By definition,

$$J_{\lambda+\beta}(t)x + \beta A_{\lambda+\beta}(t)x = \frac{\lambda}{\beta + \lambda}J_{\lambda+\beta}(t)x + \frac{\beta}{\lambda + \beta}x.$$

Now,

$$(I + \lambda A_\beta(t))\,[J_{\lambda+\beta}(t)x + \beta A_{\lambda+\beta}(t)x]$$
$$= \left(\frac{\lambda + \beta}{\beta}I - \frac{\lambda}{\beta}J_\beta(t)\right)\left(\frac{\lambda}{\beta + \lambda}J_{\lambda+\beta}(t)x + \frac{\beta}{\lambda + \beta}\,x\right) = x$$

by the resolvent identity, and this last equality if equivalent to (4.8) since $(I + \lambda A_\beta(t))$ is one-to-one for $\lambda\omega(1 - \beta\omega)^{-1} < 1$. The proof is complete.

Since $A_\beta(t)x$ is continuous in t and is Lipschitz continuous in x (see Lemma 1.2 and Lemma 4.1)

$$\begin{cases} \dfrac{du_\beta(t)}{dt} + A_\beta(t)u_\beta(t) = 0 & s \leqq t \leqq T \\ u_\beta(s) = x \end{cases}$$

has a unique classical solution for $x \in C$. If $C = X$ this is well-known, and it follows from the results of [8] in our situation. Theorem 3.1 implies $U^\beta(t,s)x = u_\beta(t)$, so $U^\beta(t,s)x$ is continuously differentiable in t. Thus, $U(t,s) = \lim_{\beta\downarrow 0} U^\beta(t,s)$ is the strong limit of C^1 evolution operators.

5. The quasi-autonomous equation

We call an initial-value problem of the form

$$\text{(5.1)} \qquad \begin{cases} \dfrac{du}{dt} + Au \ni f(t) & s \leqq t \leqq T \\ u(s) = x \end{cases}$$

quasi-autonomous. Here A is t-independent and $f: [0,T] \to X$ is single-valued. The results of Section 2 and 3 may be applied to the study of (5.1), as the next lemma shows.

LEMMA 5.1. *Let* $A \in \mathscr{A}(\omega)$, $f: [0,T] \to X$. *Let* $A(t) = A - f(t)$, $J_\lambda(t) = (I + \lambda A(t))^{-1}$, $J_\lambda = (I + \lambda A)^{-1}$, $0 < \lambda$ and $\lambda\omega < 1$. *Then* $x \in D(J_\lambda(t))$ *if and only if* $x - \lambda f(t) \in D(J_\lambda)$. *Moreover* $J_\lambda(t)x = J_\lambda(x - \lambda f(t))$ *for* $x \in D(J_\lambda(t))$ *and*

$$\text{(5.2)} \qquad \| J_\lambda(t)x - J_\lambda(\tau)x \| \leqq \lambda(1 - \lambda\omega)^{-1} \| f(t) - f(\tau) \|$$

for $x \in D(J_\lambda(t)) \cap D(J_\lambda(\tau))$.

PROOF. The assertions follow at once from the definitions and the identity $J_\lambda(t)x = J_\lambda(x - \lambda f(t))$.

The conditions (A.1), (A.2) are automatically satisfied by $A(t) = A - f(t)$ if $A \in \mathscr{A}(\omega)$, while (A.3) becomes

$$R(I + \lambda A(t)) = R(I + \lambda A) - \lambda f(t) \supseteq \overline{D(A(t))} = \overline{D(A)}$$

or

$$\text{(5.3)} \qquad R(I + \lambda A) \supseteq \bigcup_{0 \leqq t \leqq T} (\overline{D(A)} + \lambda f(t)) \quad 0 < \lambda < \lambda_0.$$

If f is continuous, then $A(t) = A - f(t)$ satisfies (C.1) due to (5.2). If f is also of bounded variation, then $A(t)$ also satisfies (C.2). Finally, it is also clear that $A(t)$ satisfies condition (ii) of Theorem 3.2 if f is continuous. We have then, as a consequence of Theorem 2.1, Proposition 2.4, Theorems 3.1, 3.2 and 3.4, the following result:

THEOREM 5.1. *Let $A \in \mathscr{A}(\omega)$ satisfy* (5.3), *where $f: [0,T] \to X$ is continuous. Then*

(i) $$U(t,s)x = \lim_{n\to\infty} \prod_{i=1}^{n} \left(I + \frac{t-s}{n}\left(A - f(s + \frac{i(t-s)}{n}\right)\right)^{-1} x$$

exists for $x \in \overline{D(A)}$ and $0 \leqq s < t \leqq T$. Moreover, $U(t,s)$ is an evolution operator on $\overline{D(A)}$.

(ii) *If $x \in \bar{D}(A)$ and u is a strong solution of* (5.1), *then $u(t) = U(t,s)x$.*

(iii) *If f is of bounded variation, then $U(t,s)$ leaves $\hat{D}(A)$ invariant and $U(t,s)x$ is Lipschitz continuous in t for $x \in \hat{D}(A)$.*

(iv) *If f is of bounded variation, A is closed and X is reflexive, then $U(t,s)x$ is a strong solution of* (5.1) *for $x \in \hat{D}(A)$.*

Theorem 5.1 (iv) extends an existence theorem of Kato [15] for quasi-autonomous equations. See also [3]. If all the requirements of (iv) do not hold, $U(t,s)x$ provides a notion of a weak or generalized solution of (5.1). It turns out that by using continuity of $U(t,s)$ as a function of $f \in L^1([0,T]: X)$ this notion can be extended further.

LEMMA 5.2. *Let g, $h: [0,T] \to X$ be two continuous functions and $A \in \mathscr{A}(\omega)$ such that* (5.3) *is satisfied with $f = g$ and $f = h$. Let U_h, U_g be the evolution operators associated with $A - h(t)$, $A - g(t)$ respectively. Then*

$$\| U_g(t,s)x - U_h(t,s)x \| \leqq \int_s^t e^{\omega(t-\tau)} \| g(\tau) - h(\tau) \| d\tau \tag{5.4}$$

for $0 \leqq s \leqq t \leqq T$, $x \in \overline{D(A)}$.

PROOF. Setting

$$\begin{aligned} a_k &= \left\| \prod_{i=1}^{k} (I + \lambda(A - g(s+i\lambda)))^{-1}x - \prod_{i=1}^{k} (I + \lambda(A - h(s+i\lambda)))^{-1}x \right\| \\ &= \left\| J_\lambda\left(\prod_{i=1}^{k-1} (I + \lambda(A - g(s+i\lambda)))^{-1}x + \lambda g(s+k\lambda)\right)\right. \\ &\quad \left. - J_\lambda\left(\prod_{i=1}^{k-1} (I + \lambda(A - h(s+i\lambda)))^{-1}x + \lambda h(s+k\lambda)\right) \right\|, \end{aligned}$$

we find that

$$a_k \leqq (1 - \lambda\omega)^{-1}(a_{k-1} + \lambda \| g(s + k\lambda) - h(s + k\lambda) \|)$$

and

$$a_n \leqq \sum_{j=1}^{n} (1 - \lambda\omega)^{j-n-1}\lambda \| g(s + j\lambda) - h(s + j\lambda) \|.$$

Set $\lambda = (t - s)/n$ above and let $n \to \infty$ to complete the proof. (The evaluation of the limit is elementary.)

Following Brezis and Benilan [1], we make the following definition:

DEFINITION 5.1. Let $A \in \mathscr{A}(\omega)$ and $g \in L^1([0, T]: X)$. Assume there is a sequence $\{f_n\}$ of continuous functions mapping $[0, T]$ to X such that (5.3) is satisfied with $f = f_n$, $n = 1, 2, \cdots$ and

$$\lim_{n\to\infty} \int_0^T \| g(\tau) - f_n(\tau) \| \, d\tau = 0. \tag{5.5}$$

Let $U_n(t, s)$ be the evolution operator associated with $A - f_n(t)$. Then

$$U(t, s)x = \lim_{n\to\infty} U_n(t, s)x \text{ for } x \in \overline{D(A)} \tag{5.6}$$

is the evolution operator associated with $A - g(t)$.

The existence of the limit (5.6) under the condition (5.5), as well as the fact that U is an evolution operator, follows at once from Lemma 5.2. Results analogous to Proposition 2.4, Theorems 3.1, and 3.4 can be established for evolution operators defined as above. We record only the following generalization of a result of [1].

LEMMA 5.3. *Let $A \in \mathscr{A}(\omega)$, $g \in L^1([0, T]: X)$ satisfy the conditions of Definition 5.1, and let $U(t, s)$ be the evolution operator associated with $A(t) = A - g(t)$. If, $x \in \overline{D(A)}$, $[x_0, y_0] \in A$, $0 \leqq s \leqq r \leqq t \leqq T$ and $\xi^* \in F(x_0 - U(r, s)x)$, then*

$$\begin{aligned}(U(r, s)x - U(t, s)x, \xi^*) \\ \leqq \int_r^t \langle y_0 - g(\tau) + \omega(x_0 - U(\tau, s)x), x_0 - U(\tau, s)x \rangle_s d\tau.\end{aligned} \tag{5.7}$$

PROOF. Let $\{f_n\}$ be the sequence in Definition 5.1 and $U_n(t, s)$ the associated evolution operators. The inequality (3.8) may be used here with $y(\tau) = y_0 - f(\tau)$ to conclude, upon reparametrizing,

$$(z - U_n(t, r)z, \xi^*) \leqq \int_r^t \langle y_0 - f_n(\tau) + \omega(x_0 - U_n(\tau, r)z), x_0 - U_n(\tau, r)z \rangle_s \, d\tau$$

for all $z \in \overline{D(A)}$ and $\xi^* \in F(x_0 - z)$. Set $z = U(r,s)x$ and let $n \to \infty$ to obtain the result. The passage to the limit is justified since $U_n(t,r)z \to U(t,r)z$ uniformly, $f_n(\tau) \to g(\tau)$ in L^1 and $\langle\ ,\ \rangle_s$ is Lipschitz continuous in its first argument as well as upper semicontinuous in both arguments.

6. Periodic solutions

In this section we are interested in the problem

$$\begin{cases} \dfrac{du}{dt} + A(t)u \ni 0 \\ u(0) = u(T) \end{cases} \tag{6.1}$$

in which the initial condition $u(s) = x$ is replaced by the "periodicity" condition $u(0) = u(T)$. The main result is:

THEOREM 6.1. *Let $\omega < 0$ and $A(t) \in \mathscr{A}(\omega)$ for $0 \leqq t \leqq T$. Further, let $A(t)$ satisfy* (A.1) *to* (A.3) *and* (C.1), *where f is of bounded variation. Let $U(t,s)$ be the evolution operator associated with $A(t)$ in the sense of Theorem* 2.1. *Then there is a unique $x_0 \in \bar{D}$ such that $U(T,0)x_0 = x_0$. Moreover, $x_0 \in \hat{D}$.*

Before proving Theorem 6.1, we note that it and Theorem 3.4 imply:

COROLLARY 6.1. *In addition to the conditions of Theorem* 6.1, *let $A(t)$ be closed for each t and let X be reflexive. Then* (6.1) *has the unique strong solution $u(t) = U(t,0)x_0$, where $U(T,0)x_0 = x_0$.*

Note that if $A(t)$ is defined for all t and periodic of period T, then $U(t,0)x_0$ is periodic of period T. Corollary 6.1 extends a similar result of [3] concerning the quasi-autonomous case in Hilbert space. See also [1].

PROOF OF THEOREM 6.1. Since $A(t) \in \mathscr{A}(\omega)$,

$$\| U(T,0)x - U(T,0)y \| \leqq e^{\omega T} \| x - y \| \qquad \text{for } x, y \in \bar{D}.$$

Since $\omega < 0$, $e^{\omega T} < 1$ and $U(T,0)$ has a unique fixed point $x_0 \in \bar{D}$. Similarly

$$P_{T/n,n} = \prod_{i=1}^{n} \left(I + \frac{T}{n} A \left(i \frac{T}{n} \right) \right)^{-1}$$

has $(1 - \omega t/n)^{-n} < 1$ as a Lipschitz constant, and has a unique fixed point $x_n \in \bar{D}$. Now

$$(6.2)\qquad \lim_{n\to\infty} P_{T/n,n}x_0 = U(T,0)x_0 = x_0$$

and

$$\begin{aligned}\| x_n - x_0 \| &= \| P_{T/n,n}x_n - x_0 \| \leqq \| P_{T/n,n}x_n - P_{T/n,n}x_0 \| + \| P_{T/n,n}x_0 - x_0 \| \\ &\leqq \left(1 - \frac{\omega T}{n}\right)^{-n} \| x_n - x_0 \| + \| P_{T/n,n}x_0 - x_0 \|,\end{aligned}$$

or

$$\| x_n - x_0 \| \leqq \left(1 - \left(1 - \frac{\omega T}{n}\right)^{-n}\right) \| P_{T/n,n}x_0 - x_0 \|,$$

so letting $n \to \infty$ and using (6.2), we have

$$(6.3)\qquad \lim_{n\to\infty} \| x_n - x_0 \| = 0.$$

In order to show $x_0 \in \hat{D}$, it is sufficient (by Lemma 1.4 and (6.3)) to show $| A(T)x_n|$ is bounded. Let

$$a_l = \left| A\left(\frac{lT}{n}\right) P_{T/n,l}x_n \right|$$

for $1 \leqq l \leqq n$. We have, via Lemma 1.3,

$$\begin{aligned} a_l &= \left| A\left(\frac{lT}{n}\right)\left(I + \frac{T}{n}A\left(\frac{lT}{n}\right)\right)^{-1} P_{T/n,l-1}x_n \right| \\ &\leqq \left\| A_{T/n}\left(\frac{lT}{n}\right) P_{T/n,l-1}\, x_n \right\| \leqq \left(1 - \frac{\omega T}{n}\right)^{-1} \left| A\left(\frac{lT}{n}\right) P_{T/n,l-1}x_n \right| \\ &\leqq \left(1 - \frac{\omega T}{n}\right)^{-1} \left\{ a_{l-1} + L(\| P_{T/n,l-1}x_n \|) \left\| f\left(\frac{lT}{n}\right) - f\left(\frac{(l-1)T}{n}\right) \right\| \right\}. \end{aligned}$$

Since $x_n \to x_0$ by (6.3), $\| P_{T/n,l-1}x_n \|$ is bounded. Thus, for a suitable K,

$$a_l \leqq \left(1 - \frac{\omega T}{n}\right)^{-1} a_{l-1} + K \left\| f\left(\frac{lT}{n}\right) - f\left(\frac{(l-1)T}{n}\right) \right\|.$$

This implies, since f is of bounded variation,

$$a_n \leqq \left(1 - \frac{T}{n}\omega\right)^{-n} a_0 + K_1$$

for another constant K_1. Hence

$$(6.4)\qquad |A(T)x_n| \leqq \left(1 - \frac{T}{n}\omega\right)^{-n} |A(0)x_n| + K_1.$$

But, by (2.3),

$$|A(0)x_n| \leqq |A(T)x_n| + K_2 \tag{6.5}$$

for a constant K_2. Since $|A(T)x_n| < \infty$ ($x_n = P_{T/n,n}x_n \in D(A(T))$), (6.4) and (6.5) give

$$|A(T)x_n| \leqq \left(1 - \left(1 - \frac{T}{n}\omega\right)^{-n}\right)^{-1} \text{Const.}$$

and it follows that

$$\limsup_{n\to\infty} |A(T)x_n| < \infty .$$

The proof is complete.

7. An example

In this section we give a simple example to which the preceding theory applies. This example is multivalued but not quasi-autonomous, so the results of [14], [15] and [3] do not apply directly.

Let Ω be a bounded domain in R^n with a smooth boundary $\partial\Omega$ and let $H^m(\Omega)$, $H_0^m(\Omega)$ be the usual Sobolev spaces. Let $\beta(t) \subset R \times R$ be a maximal monotone set in $R \times R$ for each $t \geqq 0$ (equivalently, $\beta(t) \in \mathscr{A}(0)$ and $R(I + \lambda\beta(t)) = R$ for $\lambda > 0$, $t \geqq 0$). Assume further that $D(\beta(t)) = D$ is independent of t and

$$0 \in D(\beta(t)), \quad 0 \in \beta(t)0 \text{ for } t \geqq 0. \tag{7.1}$$

The continuity condition on β will be

(C.3) There is a constant C such that if $0 \leqq t, \tau$, $x \in D(\beta(t))$ and $y \in \beta(t)x$, then there is a $w \in \beta(\tau)x$ such that

$$|y - w| \leqq C(|t - \tau|)(1 + |x|). \tag{7.2}$$

For $t \geqq 0$ we define $\bar{\beta}(t) \subset L^2(\Omega) \times L^2(\Omega)$ by

$$\bar{\beta}(t) = \{[u, v] : u, v \in L^2(\Omega) \text{ and } v(x) \in \beta(t)u(x) \text{ a.e.}\}. \tag{7.3}$$

Clearly $\bar{\beta}(t)$ is maximal monotone in $L^2(\Omega) \times L^2(\Omega)$ (equivalently, $\bar{\beta}(t) \in \mathscr{A}(0)$ and $R(I + \lambda\bar{\beta}(t)) = L^2(\Omega)$ for $\lambda > 0$, $t \geqq 0$) and $D(\bar{\beta}(t))$ is independent of t. According to (7.1) we have

$$0 \in \bar{\beta}(t)0 \text{ for } t \geqq 0. \tag{7.4}$$

THEOREM 7.1. *Let* $u_0(x) \in H^2(\Omega) \cap H_0^1(\Omega) \cap D(\bar{\beta}(t))$, *and* (7.1), (C.3) *hold. Then the initial value problem*:

$$(7.5)\qquad \begin{cases} 0 \in \dfrac{\partial u}{\partial t} - \Delta u + \bar{\beta}(t)u \text{ in } \Omega \times (0, \infty) \\ u(t,x) = 0 \qquad x \in \partial\Omega,\ t \geqq 0 \\ u(0,x) = u_0(x) \qquad \text{in } \Omega. \end{cases}$$

has a unique solution $u\colon [0,\infty) \to L^2(\Omega)$ which is Lipschitz continuous on bounded subsets of $[0, \infty)$ and such that $u(t) \in H^2(\Omega) \cap H_0^1(\Omega) \cap D(\bar{\beta}(t))$ for $t \geqq 0$.

PROOF. We will apply Theorem 3.4 in the case $X = L^2(\Omega)$. For $A(t)$ we take

$$A(t)u = -\Delta u + \bar{\beta}(t)u$$

with $D(A(t)) = H^2(\Omega) \cap H_0^1(\Omega) \cap D(\bar{\beta}(t))$. Clearly $A(t) \in \mathscr{A}(0)$ for $t \geqq 0$. Moreover $R(I + \lambda A(t)) = L^2(\Omega)$ for $\lambda > 0$. See [4]. This implies that each $A(t)$ is closed in $L^2(\Omega) \times L^2(\Omega)$. Hence (A.1), (A.2), (A.3) are satisfied by $A(t)$. To use Theorem 3.4 we show (C.1) is satisfied with f of bounded variation (so (C.2) is satisfied). Let $h \in L^2(\Omega)$ and

$$\begin{cases} u_1 - \lambda\Delta u_1 + \lambda v_1 = h \qquad v_1 \in \bar{\beta}(t)u_1 \\ u_2 - \lambda\Delta u_2 + \lambda v_2 = h \qquad v_2 \in \bar{\beta}(\tau)u_2. \end{cases}$$

Forming the difference of these equalities, multiplying by $u_1 - u_2$ and integrating over Ω yields

$$(7.6)\qquad \| u_1 - u_2 \|^2 + \lambda \int_\Omega (v_1(x) - v_2(x))(u_1(x) - u_2(x))\,dx \leqq 0.$$

Next, let $w\colon \Omega \to R$ be such that $w(x) \in \beta(t)u_2(x)$ a.e. and

$$| w(x) - v_2(x) | \leqq C |t - \tau| \,(1 + |u_2(x)|) \text{ a.e.}$$

The existence of w follows from (C.3). Since $\beta(t)$ is monotone in $R \times R$, we have

$$\begin{aligned} (v_1(x) - v_2(x))(u_1(x) - u_2(x)) &= (v_1(x) - w(x) + w(x) - v_2(x))(u_1(x) - u_2(x)) \\ &\geqq (w(x) - v_2(x))(u_1(x) - u_2(x)) \\ &\geqq -C|t - \tau|\,(1 + |u_2(x)|)(u_1(x) - u_2(x)| \end{aligned}$$

a.e. in x. Using this in (7.6) yields

$$(7.7)\qquad \begin{aligned} \| u_1 - u_2 \|^2 &\leqq \lambda C |t - \tau| \int_\Omega (1 + |u_2(x)|)(|u_1(x) - u_2(x)|)\,dx \\ &\leqq \lambda C |t - \tau| (\mu(\Omega)^{\frac{1}{2}} + \| u_2 \|)(\| u_1 - u_2 \|). \end{aligned}$$

where μ is Lebesgue measure. Finally, $u_2 = J_\lambda(\tau)h$, $u_1 = J_\lambda(t)h$ by definition and $J_\lambda(\tau)0 = 0$ by (7.4) Hence

$$\| u_2 \| = \| J_\lambda(\tau)h - J_\lambda(\tau)0 \| \leqq \| h \|$$

and (7.7) yields

$$\| J_\lambda(t)h - J_\lambda(\tau)h \| \leqq \lambda C | t - \tau | \, (\mu(\Omega)^{\frac{1}{2}} + \| h \|). \tag{7.8}$$

Thus (C.1) and (C.2) are satisfied. (Observe that weakening (C.3) by putting $|f(t) - f(\tau)|$ in place of $| t - \tau |$ induces the same alteration in (7.8)). The proof is complete.

A simple $\beta(t)$ satisfying all our assumptions is

$$\beta(t)x = \begin{cases} [-t, t] & \text{if } x = 0 \\ \{t(1+x)\} & \text{if } x > 0 \\ \{t(x-1)\} & \text{if } x < 0. \end{cases}$$

Since $A = -\Delta$ on $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ belongs to $\mathscr{A}(-\mu_0)$ where $\mu_0 > 0$ is the smallest eigenvalue of $-\Delta$, we actually have $A(t) \in \mathscr{A}(-\mu_0)$ above. Hence we can apply Corollary 6.1 to obtain periodic solutions of the evolution equation.

THEOREM 7.2. *Let $T > 0$ and the assumptions of Theorem 7.1 hold. Then the problem*

$$\begin{cases} 0 \in \dfrac{\partial u}{\partial t} - \Delta u + \bar{\beta}(t)u & x \in \Omega, \quad t \geqq 0 \\ u(t,x) = 0 & x \in \partial\Omega, \quad t \geqq 0 \\ u(0,x) = u(T,x) \end{cases}$$

has a unique solution $u: [0,\infty) \to L^2(\Omega)$, which is Lipschitz continuous on bounded subsets of $[0,\infty)$ and satisfies

$$u(t) \in H^2(\Omega) \cap H_0^1(\Omega) \cap D(\bar{\beta}(t)) \text{ for } t \geqq 0.$$

If $\bar{\beta}(t)$ has period T, so does u.

Appendix 1

Here we sketch a proof by induction of inequality (2.19) of the text.

LEMMA A. *Let m and n be nonnegative integers and $a_{k,l}$, $b_{k,l}$ be real numbers for $0 \leqq k \leqq m$, $0 \leqq l \leqq n$. Let γ and κ be real numbers such that*

$$a_{k,l} \leqq \gamma a_{k-1,l-1} + \kappa a_{k,l-1} + b_{k,l} \tag{a.1}$$

for $0 < k \leqq m$, $0 < l \leqq n$. *Then*

$$\begin{aligned} a_{m,n} \leqq & \sum_{i=0}^{(m-1)\wedge n} \kappa^{n-i}\gamma^i \binom{n}{i} a_{m-i,0} + \sum_{i=m}^{n} \gamma^m \kappa^{i-m} \binom{i-1}{m-1} a_{0,n-i} \\ & + \sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \kappa^{j-i}\gamma^i \binom{j}{i} b_{m-i,n-j}. \end{aligned} \tag{a.2}$$

PROOF. We use the conventions $0^0 = 1$, $\binom{-1}{-1} = 1$ and $\binom{j}{-1} = 0$ if $j \geqq 0$. If $n = 0$ (respectively, $m = 0$) (a.2) asserts $a_{m,0} \leqq a_{m,0}$ (respectively, $a_{0,n} \leqq a_{0,n}$), which is correct. If $m = 1$, (a.2) says

$$a_{1,n} \leqq \kappa^n a_{1,0} + \sum_{i=1}^{n} \gamma\kappa^{i-1} a_{0,n-i} + \sum_{j=0}^{n-1} \kappa^j b_{1,n-j}. \tag{a.3}$$

As noted above, (a.3) is correct for $n = 0$. Assume (a.3) holds for $0 \leqq n \leqq N$. Then (a.1) gives

$$a_{1,N+1} \leqq \gamma a_{0,N} + \kappa a_{1,N} + b_{1,N+1}. \tag{a.4}$$

Use the induction assumption to conclude

$$\begin{aligned} a_{1,N+1} \leqq & \gamma a_{0,N} + \kappa \Bigg(\kappa^N a_{1,0} + \sum_{i=1}^{N} \gamma\kappa^{i-1} a_{0,N-i} \\ & + \sum_{j=0}^{N-1} \kappa^j b_{1,N-j}\Bigg) + b_{1,N+1} \end{aligned} \tag{a.5}$$

which is precisely (a.3) for $n = N + 1$. Hence the lemma is true if $m = 0$ or $n = 0$ or $m = 1$. To complete the proof, we assume the lemma is true if $0 \leqq n \leqq N$ and m is arbitrary. It now suffices to show (a.2) holds if $m \geqq 2$ and $n = N + 1$. (The reduction to $m \geqq 2$ eliminates concern over the expressions $\binom{i-1}{m-2}$ below.)

From (a.1) we have, for $m \geqq 2$,

$$a_{m,N+1} \leqq \gamma a_{m-1,N} + \kappa a_{m,N} + b_{m,N+1}.$$

Using the induction hypothesis we obtain

$$
\begin{aligned}
a_{m\,N+1} \leqq & \left[\gamma \sum_{i=0}^{(m-2)\wedge N} \kappa^{N-i}\gamma^{i}\binom{N}{i} a_{m-(i+1),0}\right. \\
& \left. + \kappa \sum_{i=0}^{(m-1)\wedge N} \kappa^{N-i}\gamma^{i}\binom{N}{i} a_{m-i,0}\right] \\
& + \left[\gamma \sum_{i=m-1}^{N} \gamma^{m-i}\kappa^{i+1-m}\binom{i-1}{m-2} a_{0,N-i}\right. \\
& \left. + \kappa \sum_{i=m}^{N} \gamma^{m}\kappa^{i-m}\binom{i-1}{m-1} a_{0,N-i}\right] \\
& + \left[\gamma \sum_{j=0}^{N-1} \sum_{i=0}^{(m-2)\wedge j} \kappa^{j-i}\gamma^{i}\binom{j}{i} b_{m-(i+1),N-j}\right. \\
& \left. + \kappa \sum_{j=0}^{N-1} \sum_{i=0}^{(m-1)\wedge j} \kappa^{j-i}\gamma^{i}\binom{j}{i} b_{m-i,N-j} + b_{m,N+1}\right].
\end{aligned} \tag{a.6}
$$

Now one checks each of the three terms in brackets above to verify that it agrees with the corresponding term on the right of (a.2) (with $n = N + 1$). This is straightforward for the first and second terms. The third term may be rewritten as

$$
\begin{aligned}
b_{m,N+1} & + \sum_{k=1}^{N} \sum_{l=1}^{(m-2)\wedge(k-1)+1} \kappa^{k-l}\gamma^{l}\binom{k-1}{l-1} b_{m-l,N+1-k} \\
& + \sum_{k=1}^{N} \sum_{l=0}^{(m-1)\wedge(k-1)} \kappa^{k-l}\gamma^{l}\binom{k-1}{l} b_{m-l,N+1-k}.
\end{aligned}
$$

Note that

$$
(m-2)\wedge(k-1)+1 = \begin{cases} (m-1)\wedge(k-1) & k \geqq m \\ (m-1)\wedge(k-1)+1 & 0 \leqq k < m. \end{cases}
$$

Now reading off the coefficient of $b_{m-l}(_{N+1})_{-k}$ above, case by case, yields

$$
1 = \kappa^{k-l}\gamma^{l}\binom{k}{l} \quad \text{for} \quad l = k = 0
$$

$$
\kappa^{k-l}\gamma^{l}\binom{k-1}{l} = \kappa^{k-l}\gamma^{l}\binom{k}{l} \quad \text{if} \quad l = 0, \quad k > 0
$$

$$
\kappa^{k-l}\gamma^{l}\left(\binom{k-1}{l-1} + \binom{k-1}{l}\right) = \kappa^{k-l}\gamma^{l}\binom{k}{l} \quad \text{if} \quad (m-1)\wedge(k-1) \geqq l > 0
$$

and

$$
\kappa^{k-l}\gamma^{l}\binom{k-1}{l-1} = \kappa^{k-l}\gamma^{l}\binom{k}{l} \text{ if } (m-2)\wedge(k-1)+1 = l > (m-1)\wedge(k-1)
$$

(since then $l = k$).

Each pair of integers k, l such that $0 \leqq l \leqq (m-1) \wedge k$ and $0 \leqq k \leqq N$ falls into exactly one of the cases above, and the proof is complete.

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